UNIVERSIDAD CARLOS III DE MADRID

TESIS DOCTORAL

Marketing Strategies for Enhancing Word of Mouth: Sales Acceleration and Affiliate Programs

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DEPARTAMENTO DE ECONOMÍA DE LA EMPRESA

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Abstract

For years, marketing practitioners and scholars have acknowledged consumers' Word Of Mouth (WOM) as a key driving force behind the success of new products and technological innovations, and more generally for an effective firm communication over time. Marketing managers have developed WOM-marketing tools to take better advantage of buzz.

This thesis is comprised of three essays on WOM marketing in a dynamic context that consider several strategies that a company can use to enhance WOM and to accelerate the diffusion of new products from a managerial perspective. We also discuss relatively recent WOM online communication tools, such as affiliate marketing, their effectiveness and dynamic effects.

The main contributions of this thesis can be summarized as follows:

- Inventors can commercialize innovative products by themselves and simultaneously license the technology to other firms. The licensee may cannibalize sales of the licensor, but this can be compensated by gains from royalties. In the first essay of this thesis, we show how licenses can be used strategically to speed up the new product diffusion process in two instances of markets: (i) a market with strong Intellectual Property Rights (IPR), and (ii) a market with weak IPR holder and pirate rivals. The main findings suggest that licensing is a beneficial strategy for a licensor in the context of strong IPR, because licensor benefits from the royalties, the advertising investment and positive word-of-mouth effects by licensees. We compare this result with a weak IPR context, where piracy speeds up the product diffusion but this does not compensate IPR holder for the sales loss effect who is willing to license to get some royalties. However, pirates do not generally find interesting the licensing agreement. We present a comparative statics analysis based on numerical simulation. We illustrate the application of the proposed licensing model to incandescent light bulbs industry in the United Kingdom.

- Managing diffusion waves for successive product generations implies that marketing managers should try persuade some customers to swim over the successive generation waves optimally for the company profits, as well as to lessen the regret of old-generation product buyers. In the second essay we discuss how trade-in rebates can be used to reintegrate owners of old versions of the product to the market and therefore accelerate current sales. We first build a general diffusion model for successive product generations. We study the optimal behavior of the firm controlling the prices and rebates associated to product upgrades. In order to quantify the effect of upgrade-rebate strategy, we particularize the general model for some concrete examples and numerically solve them for certain sets of parameter values. We demonstrate that the trade-in program accelerates the diffusion of the later generations but has the reverse effect for the diffusion of the old generation product. The size of the percentage gain in profits varies depending on several conditions, providing a 2-5% increase in total discounted profits. We illustrate the applicability of the model for automobile industry in Spain for the period from 1970 to 2000.

- In the third essay of the thesis, we focus on the analysis of the effectiveness of affiliate marketing. Many online customers who visit a retailer's website through affiliate companies may later return to the retailer for a subsequent purchase through other web traffic sources. These customers might refer other potential online shoppers through word of mouth effect. On the other hand, affiliate companies might cannibalize the retailer's other marketing effort. In the third essay we study the dynamic effect of affiliate marketing on the advertiser's traffic, sales and revenues. Because affiliates vary in the volume of their operations, contribution to the advertiser's online sales, in their marketing tools and strategies, the effect of affiliates on advertiser's sales and revenues is likely to be heterogeneous across affiliates. Given the large number of affiliates, estimation using standard VAR or VEC Models is challenging because of large dimensions involved. Instead, we employ GVAR analysis introduced by Pesaran et al (2004). We present an empirical application

based on data from an online retailer of jewelry. Our findings reflect in detail the dynamic forces shaping the affiliate marketing industry. The results of Impulse Response Function analysis show significant long term effect of affiliate marketing that clearly differs across affiliates.

Resumen en Español

Durante años, los profesionales y académicos dedicados al marketing han identificado la estrategia del "marketing boca a boca" (denotada por el acrónimo anglosajón WOM por Word-of Mouth) como una fuerza impulsora clave para el éxito empresarial. Los gerentes de marketing deben tener en cuenta la presencia y el impacto del marketing boca-de-boca para las empresas y cómo las empresas pueden aprovechar mejor los efectos de esta. Esta tesis se compone de tres ensayos sobre el marketing boca a boca en un contexto dinámico que evalúa cómo una empresa puede beneficiarse de esta estrategia puesto que tiene un efecto de aceleración de sus ventas y si las estrategias de WOM de comunicación, como el marketing de afiliados, resultan estrategias atractivas para la empresa.

Las principales contribuciones de esta tesis pueden resumirse así:

- Los inventores pueden comercializar ellos mismos productos innovadores y simultáneamente conceder licencias sobre la tecnología a otras firmas. El licenciatario o licenciado (o entidad que adquiere la licencia) puede canibalizar las ventas del licenciador (o titular de una licencia) pero este efecto puede verse compensado por las ganancias provenientes de las regalías o canon por disfrute de los derechos de la licencia. En el primer ensayo mostramos cómo se pueden usar estratégicamente las licencias para comercialización de nuevos productos para acelerar su proceso de difusión en dos tipos de mercado: (i) un mercado donde los derechos de propiedad intelectual están fuertemente protegidos, y (ii) un mercado con derechos de propiedad donde éstos no están tan protegidos y empresas rivales comercializan versiones piratas. Los resultados obtenidos muestran que, en presencia de mercados fuertemente protegidos, la concesión de licencias es una estrategia conveniente para un licenciador (que se beneficia de los cánones por cesión, la inversión en publicidad y el efecto positivo del boca a boca entre los licenciatarios). Comparado este resultado con el obtenido en un mercado menos protegido, observamos que observando quela piratería acelera la difusión de los productos pero el licenciante sufre una pérdida de ventas que no se ve compensada de ninguna forma ya queen general los piratas no están interesados en el acuerdo de licencia y no se paga ningún canon. El estudio finaliza con un análisis de estática comparativa basado en simulaciones numéricas, y un estudio empírico del modelo de licencias en la industria de lamparas incandescentes en Reino Unido.

- Cuando se introducen nuevas generaciones de un producto duradero, la superposición de ventas de cada una de ellas recuerdan una secuencia de olas. En este capítulo se considera una estrategia para la dirección de marketing ante las distintas oleadas de nuevas , motivando a los clientes de las primeras generaciones para que se pasen a las de nueva generación. Este trasvase implica mayores beneficios para la empresa y una mejora de la satisfacción de los clientes de productos de vieja generación. En concreto, se estudia cómo las promociones de canje por cambio de generación pueden usarse para convencer a los consumidores de versiones antiguas del producto pasarse a versiones más actualizadas y, de paso, acelerar las ventas de productos más actuales. Para ello, primero construimos un modelo general de difusión para generaciones sucesivas de productos. Después, estudiamos el comportamiento de decisión óptimo de la empresa controlando por los precios y las promociones asociadas a la actualización de productos. Con el fin de cuantificar el efecto de las promociones (descuentos en la actualización del producto), construimos diversos modelos que representan distintas situaciones del mercado, y los resolvemos numéricamente para ciertos valores de los parámetros. Demostramos que los descuentos por actualización de la generación del producto aceleran la difusión de las últimas generaciones pero presentan el efecto contrario en la difusión de los productos de la vieja generación. El tamaño de la ganancia porcentual en los beneficios depende de varias condiciones, presentándose un aumento entre el 2 y el 5% en total de los beneficios descontados. Además, mostramos la aplicabilidad del modelo propuesto en la industria de autómobiles en España en el periodo de 1970 a 2000.

- El tercer capítulo de la tesis se centra en el estudio del marketing de afiliados. Muchos de los clientes virtuales que visitan la página web de un minorista a través de compañías afiliadas probablemente volverán al minorista para realizar una compra subsecuente a través de otras fuentes de tráfico en línea. Estos clientes pueden referenciar a otros compradores virtuales por medio del boca a boca. Por otra parte, las compañías afiliadas pueden canibalizar el resto del esfuerzo de marketing del minorista. En el tercer ensayo estudiamos el efecto dinámico del marketing por afiliación en el tráfico, las ventas y los ingresos del anunciante. Puesto que los afiliados difieren en el volumen de operaciones, en la contribución a las ventas en línea, en sus herramientas y estrategias de marketing, su efecto en las ventas y los ingresos del anunciante probablemente es heterogéneo. Dado el gran número de afiliados, la estimación usando modelos VAR o VEC es un problema intratable debido a las dimensiones involucradas. En su lugar, usamos la teoría GVAR propuesto por Peseran et al (2004). Presentamos una aplicación empírica basada en un minorista de joyería on-line. Nuestros resultados muestran en detalle las fuerzas dinámicas que dan forma a la industria del marketing por afiliación. Los resultado del análisis basados en la función de impulso-respuesta muestran que el efecto de largo plazo del marketing por afiliación es significativo y difiere claramente entre afiliados.

Contents

Chapter 1

Introduction

1.1 The Power of Word of Mouth

The phenomenon of word of mouth has been acknowledged for several years. A large body of academic studies, industry market research and empirical evidence have shown the important role of word of mouth in consumer behavior and firm actions. Arndt (1967) was one of the earliest scholars in marketing to characterize word of mouth as oral, person-to-person communication between a receiver and a communicator whom the receiver perceives as non-commercial, regarding a brand, product or a service. However, in digital age word of mouth does not have to be necessarily face-to-face, direct, or oral, and could be also focused on an organization and not only a brand or product (Buttle, 1998).

Several articles have been published about word of mouth covering numerous topics. In the context of new product introduction, word of mouth is of crucial importance for the diffusion of innovation. The diffusion of innovations is usually defined as the dynamic process by which the information about a social innovation (e.g. a product or service) is communicated to the members of the social system through mass media and interpersonal channels (Rogers, 1995). Product innovation diffusion models have a long history in marketing, but the most successful model was proposed in a seminal paper by

Bass (1969). Mahajan et al (1990, 1993), Chandrasekaran and Tellis (2007) and more recently, Peres et al (2010) provide critical review of diffusion models, while Sultan et al. (1990) conduct meta-analysis of diffusion models.

A large number of researchers have studied different marketing mix decisions accounting for word of mouth effect – advertising (e.g. Horsky and Simon, 1983; Horsky and Mate, 1988), price (e.g. Robinson and Lakhani, 1975; Kalish, 1985; Krishnan et al. 1999; Bass et al. 1994), shelf space (e.g. He and Sethi, 2008), etc. For the case of advertising, word of mouth generates a "ripple effect", making the effect of advertising long-lasting (Monahan, 1984; Hogan et al, 2004).

Taking a different perspective, other scholars have studied the innovation adoption decision by individuals (e.g. Horsky 1990; Weerahandi and Dalal 1992), as well as the heterogeneity in the communication behavior of adopters (e.g. Goldenberg et al. 2001). Using individual adoption model, Ho et al. (2012) study how firms can control the social contagion process and actively influence on the diffusion acceleration. Word of mouth has also been studied in the context of multiple product generations diffusion (Norton and Bass, 1987; Mahajan and Muller, 1996; Jiang and Jain, 2012), different segments of adopters (Mahajan and Muller, 1998; Goldenberg et al, 2002; Lehmann and Esteban-Bravo, 2006), among others. Other studies have attempted to quantify the effect of word of mouth on firm performance (e.g. Luo, 2009; Trusov et al. 2009).

Hence, marketing managers should account for the presence of word of mouth effect, harness the power of word of mouth and optimally utilize it for the company's profits. In this thesis, employing optimal control theory and applied econometrics techniques, in three different essays we show how a company could benefit from the power of word of mouth by accelerating its sales. We also shed light on the effectiveness of an online advertising tool, such as affiliate marketing, taking into account the word of mouth effect.

1.2 Dissertation Outline

The different chapters of the thesis all reflect the importance of word of mouth in three different marketing contexts: commercialization of radical product innovation, management of multiple product generation diffusion, and affiliate marketing. This section provides an overall outline of the dissertation.

First Essay: Licensing Radical Innovations

In the first chapter of the dissertation we investigate how licensing can be used to speed up the diffusion of new products. Licensing an innovation implies that the inventor sacrifices with its market share to the licensee companies. This research study considers sales diffusion process and looks at the benefits of licensing as a strategy to improve the licensor's profits by speeding up the sales diffusion through advertising investments and word-of-mouth effects of licensee companies.

The initial diffusion process of radical innovations is usually characterized by a slow growth that is afterwards followed by a large increase known as sales "takeoff" (e.g., Mahajan et al. 1990, Rogers 1995, Golder and Tellis 1997). The take-off time and the speed of the diffusion are critical for companies with deep implications over the supply-chain, inventory and product distribution management. It has also a crucial impact on firm value (an early takeoff increases the net present value of the innovation, as revenues cashed into the distant future are heavily discounted). Thus, there has been much interest to study the take-off time and the speed of the diffusion. The time to takeoff in the sales diffusion of radical product innovations can vary considerably (e.g., Mahajan et al. 1990, Golder and Tellis 1997). There are also demand cultural factors suggesting that sales takeoff can vary over different countries (Tellis, Stremersch and Yin 2003).

Most of the managers favor faster diffusion of the products which they launch in the market. Previous literature has established that marketing mix factors, particularly price-decreases and advertising effort, can partially explain the takeoff times and diffusion speed (e.g., Stoneman and Ireland 1983, Golder and Tellis 1997, 2004, Foster et al. 2004). In addition, Agarwall and Bayus (2002) found that the entry of new competitors during the early years of the market can push the demand outward, driven by improvements in product quality, distribution infrastructures, and higher awareness, suggesting that firm entrance may dominate the classical marketing-mix factors in explaining the takeoff times. In the first essay we consider licensing as a strategy to accelerate sales diffusion. Competition between the licensor and the licensees speeds up the sales of the newly commercialized product due to more innovation through combined advertising effort and cross-word of mouth effect. As a consequence, for a monopolistic Intellectual Property Rights (IPR) holder sales acceleration effect and revenue effect due to licensing fixed fees and royalties outweigh rent dissipation effect due to sacrifice of market share to the licensees.

An analytical dynamic model is presented, featuring the licensor-licensee behavior as an open-loop Nash equilibrium in a differential game. We show that the consequences of this strategy are different in the context of strong and weak IPR protection. We obtain optimal policies of marketing mix and licensing fees in both IPR contexts. To get in-depth insight of licensing strategy, we also provide a sensitivity analysis to the model's parameters based on numerical simulations. Based on the analysis of the dynamic model and comparative statics, we provide several managerial implications for a firm considering licensing as a strategy to get faster sales. As a final step, we provide an empirical application of the proposed licensing model to incandescent light bulbs industry in the United Kingdom.

Second Essay: Upgrade-Rebate Programs

In the second essay we show how trade-in rebates could be used to manage technological diffusion waves. Analyzing the trade-in strategy in a dynamic context, we investigate how it can be used to control the whole diffusion of the successive technological generations accelerating the diffusion process of later generations through word of mouth effect of upgraders.

In the context of multiple product generation diffusion, there are several issues present. The older generations of the product are usually perceived as of worse quality than the new generations, and thus firms lower their prices (Koenigsberg and Ferguson, 2007). Sales of different generations cannibalize each other while firms sell them simultaneously. Further, the fear of obsolescence may cause some of the customers to refuse buying the earlier generations and wait for the new generation releases (Cohen et al. 1996; Dhebar 1996; Venkatesh and Brown, 2001). Although second hand markets might reduce this threat of leapfrogging, they also create additional competition for the firm (Desai, Koenigsberg, and Purohit, 2007).

Most of the previous models analyze rebate programs in static context. Previous research considered trade-ins as a strategic link between reverse and direct channels management (Ray, Boyaci, and Aras, 2005), as a price discrimination policy (Lee and Lee, 1998), as a tool to disable second hand market (Levinthal and Purohit, 1989), as a means to increase purchase frequency of quasi-durable goods (Ackers and Reyniers, 1995). We take a different approach analyzing the trade-in strategy in a dynamic context, where we show how the company could create a tsunami of a new generation through an upgrade-rebate program due to word of mouth acceleration effect of the upgraders.

Building on previous work regarding of technology generation diffusion (Norton and Bass, 1987, 1992, Mahajan and Muller 1996, and Stremersch, Muller and Peres 2010), we first present a general diffusion model for successive K generations of durable product each generation with its own market potential. The model allows for leapfrogging, can be applied to study diffusion of non-durable and semi-durable products, and can be easily adapted to analyze upgrade-rebate program. In the model where rebate programs are present, those customers who trade-in, generate word of mouth affecting the diffusion of product generations.

We then particularize the general model for some concrete examples (two and three product generations, inclusion of prices as decision variable, free rebates). We utilize tools of optimal control theory (Kamien and Schwartz, 1991) and numerical methods to solve for firm's optimal decision problem for a certain set of parameters. For all the cases we show that trade-ins accelerate the diffusion of subsequent generations but have the reverse effect on the diffusion of the old generation products. In order to quantify the effect of the program, we compute the optimal discounted profits under scenarios of implementing upgrade-rebate strategy or not. We find that the implementation of the upgrade-rebate strategy provides a 2-5% increase in total discounted profits.

We also find that for the firm it is optimal to provide to the upgraders rebate amounts even higher than the price of the new generation up to some time. This is in line with the findings of studies in the context of one product generations about subsidizing adopters and product give aways (e.g. Kalish and Lilien 1983, Lehmann and Esteban-Bravo 2006). Finally, using cars sales and scrappages we illustrate the applicability of the model for automobile industry in Spain for the period from 1970 to 2000.

Third Essay: Affiliate Marketing

The third essay of the dissertation studies the dynamic performance of affiliate marketing. Using Global Vector Auroregression (GVAR) methodology, we investigate and quantify the indirect and dynamic effect of affiliate marketing on the advertiser's traffic, sales and revenues. We also examine the long-term effect of each affiliate's traffic on sales quantity and dollar amount provided to the advertiser.

Affiliate marketing is an important source of customer acquisition of many online retailers (Hoffman and Novak, 2000; Libai et al, 2003). In fact, Forrester Research estimates that affiliate marketing spending in the United States will reach $4 billion by 2014, with annual growth rate of 16%. Affiliate marketing program consists of an online retailer placing a link on a third party (i.e. affiliates') website. The advertiser pays the affiliates a referral fee for every referral that has been converted into a buyer (pay-per-conversion). Another commonly used mechanism is called pay-per-lead, when the payment is done regardless whether the referrals are converted into buyers or not (Libai et al., 2003).

Only immediate visits and generated revenues are attributed to affiliate company. However, we propose that there are also dynamic effects of affiliate marketing which are not reflected in the traditional managerial metrics (e.g. click through rate, conversion rate, etc.). These dynamic effects are due to word of mouth effect, since some of the customers may refer other potential buyers to the website. Also, the customers acquired by affiliates may later return to the online retailer's website through other channels of traffic for a subsequent purchase which should be attributed to affiliate marketing. On the other hand, affiliates may cannibalize the merchant's other marketing effort (i.e. they refer those customers who would visit the retailer's website in any case).

In an empirical application with data from an online retailer of jewelry, we study the impact of affiliate marketing on the advertiser's performance. Affiliate companies vary in the volume of their operations, contribution to the advertiser's online sales (Ray, 2001), marketing strategies, website design, etc. Thus, we posit that the effect of affiliates on advertiser's sales and revenues is heterogeneous across affiliates. We quantify the effect of affiliate marketing on merchant's sales and revenues using recent advances in the analysis of dynamic factor models.

The remainder of the dissertation is structured as follows. In the next chapter we present the essay entitled "Licensing Radical Product Innovations to Speed up the Diffusion". The third chapter deals with the essay entitled "Riding Successful Technological Diffusion Waves: Building a Tsunami via Upgrade-Rebate Programs". In the fourth chapter we focus on the essay entitled "The Dynamic Performance of Affiliate Marketing".

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Chapter 2

Licensing Radical Product Innovations to Speed Up the Diffusion

2.1 Introduction

Instead of commercializing the innovation alone, an inventing firm can license the product technology to one or more other firms. A license is a contract by which an IPR holder firm (licensor) transfers the right to exploit its innovation to another firm (licensee) under certain conditions and for a certain period of time. Licensing generates two forces on licensor's profit: *revenue effect* (licensing payments by the licensees to the IPR holder) and *rent dissipation effect* (erosion of licensor's profit due to additional competition in the product market). On a first look, licensing IPR is a daring decision as the rent dissipation effects might be stronger than the revenue one. The early literature on licensing is focused on static models (for a review see, e.g., Shapiro 1985 and Kamien 1992). More recently, Arora and Fosfuri (2003) develop a framework to understand licensing and competition. Indeed, a monopolist IPR holder will not license as the rent dissipation effect is greater than the revenue effect. Moreover, in practice, we observe that royalties are often low, and licensors capture only a small fraction of the rents from the innovation (e.g., Caves et al. 1983, Arora 1997). Arrow (1962) remarks show how striking this feature is: "Patent royalties are generally so low that the profits from exploiting one's own invention are not appreciably greater than those derived from the use of others' knowledge". Why the firm that has developed the knowledge cannot demand a greater share of the resulting profits? Surprisingly, licensing is a pervasive phenomenon. According to the *License! Global* 2008 Annual Report, the total worldwide retail sales of licensed merchandize reached $191.7 billion in 2007. How can this contradiction be explained?

A potential solution to this conundrum could lie hidden in the dynamics of innovation adoption. The diffusion of new products is typically modeled with first order differential equations where the solution is an "S" shape curve. After commercialization, the early diffusion of radical innovations is usually characterized by a slow growth that is eventually followed by a sharp increase known as sales "takeoff" (e.g., Mahajan et al. 1990, Rogers 1995, Golder and Tellis 1997). Diffusion takeoff time and speed are critical for the company, with deep implications over the supply-chain, inventory and product distribution management. It has also a crucial impact on firm value (an early takeoff increases the net present value of the innovation, as revenues cashed into the distant future are heavily discounted). The time to takeoff in the sales diffusion of radical product innovations can vary considerably (e.g., Mahajan et al. 1990, Golder and Tellis 1997). There are also demand cultural factors suggesting that sales takeoff can vary on different countries (Tellis, Stremersch and Yin 2003). Over the last decade, there has been much interest in explaining sales takeoff of product innovation diffusion. The literature is mostly descriptive and has established that marketing mix factors, particularly price-decreases and advertising effort, can partially explain the takeoff times (e.g., Stoneman and Ireland 1983, Golder and Tellis 1997, 2004, Foster et al. 2004). In addition, Agarwal and Bayus (2002) found that the entry of new competitors during the early years of the market can push the demand outward, driven by improvements in product quality, distribution infrastructures, and higher awareness, suggesting that firm entrance may dominate the classical marketing-mix factors in explaining the takeoff times. For incremental innovations, there is also some evidence of cross-generation acceleration (Stremersch et al. 2010). In addition, competition can benefit market size through advertising (Roberts and Samuelson 1988). Loosely speaking, the affluence of competing firms seems to spur higher innovation awareness through combined advertising and promotional efforts, price reductions due to firm rivalry, and product differentiation by quality improvements that (moderated by socio-demographic and environmental factors) can explain the first large increase in sales. But innovation ownership is usually protected by the Intellectual Property Rights (IPR) generating a temporal grant of monopoly power over the right to make commercial use of ideas. This protection may prevent the entrance of other firms, and therefore delay the takeoff time and/or decrease the diffusion speed.

This paper considers an alternative reason to license: *sales diffusion acceleration*. This third effect is neglected in static models of licensing. Competition between licensor and licensees results in faster sales diffusion due to higher innovation awareness through the combined marketing effort and cross word-of-mouth effects. As a result, for a monopolistic IPR holder *sales diffusion acceleration* and *revenue effects* dominate *rent dissipation effect* (loss of the market to the licensees), and licensing takes place. The marketing literature supports this idea. Armstrong and Collopy (1996) and Luo, Rindfleisch and Tse (2007) argue that competitor-oriented decisions such as exclusivity are harmful to financial performance. Recently, Peres and Van den Bulte (2010) discuss that word-of-mouth turns product monopoly suboptimal.

The success of a licensing strategy depends on the strength of the IPR system. Several studies have empirically considered the relationship between patent protection and licensing, finding that there is a higher propensity to license in industries with strong patent protection (e.g., Anand and Khanna 2000, Arora and Ceccagnoli 2006, Gambardella et al. 2007). We study the use of licenses as a strategy to speed up new products diffusion when the IPR holder operates in: (i) a market with strong IPR, and (ii) a market with weak IPR and pirate rivals, who commercialize unlicensed product imitations. For each framework, an analytical dynamic model is presented, featuring the Licensor-Licensee behavior as an open-loop Nash equilibrium in a differential game.

- In markets with strong IPR protection, our results show that both IPR holders and licensees can benefit from licensing. For the IPR holder licensing is a beneficial strategy because there is an increase in profits due to the acceleration of the sales diffusion process, in comparison to the case of monopoly. In comparative statics we also analyze how sensitive licensing implementation is to the model parameters. This result stands in contrast to the results drawn from static licensing models.

- Regarding industries with weak IPR protection, we study licensing decision in markets where IPR holder faces weak competition by pirate companies who sell copy products with lower quality. From IPR holder's perspective, licensing is a beneficial decision due to licensing payments and faster sales diffusion. However, pirates are better off when selling copy products than licensed products with higher quality as in the last case they have to pay licensing fees to the IPR holder. Although not completely ruled out, comparative statics suggests that licensing is less desirable strategy in the context of weak IPR protection from either the perspective of the IPR holder or the pirates. This can partially explain, for instance, the slowness of discography industry to allow licensing through the internet to stop the boom of piracy in the last decade. Pirates are less interested in this arrangement. This is supported by the empirical evidence indicating that licensing is less common in this context (Anand and Khanna 2000, Arora and Ceccagnoli 2006, Gambardella et al. 2007).

In both cases, we have also obtained firms' optimal policies for marketing mix and licensing fees in both of IPR contexts. Interestingly, in strongly protected IPR framework the discounted prices charged by monopolistic, licensor and licensees are not too different. By contrast, in weakly protected IPR context there are higher differences between the average prices (which can be explained by the higher independence between the diffusion process of pirates and IPR holder). In the case of licensing, optimal fixed fees as well as royalties decrease to zero exponentially. The discounted advertising investments decrease at an exponential rate in all the cases.

The remainder of the paper is structured as follows. In the next section we present the monopolistic innovation diffusion model for non-durable products that we use as a benchmark in the paper. In section 3 we characterize optimal licensing, pricing and advertising strategies and with numerical methods analyze the sensitivity of the optimal profits to the main parameters, when IPR are strong. In section 4, we conduct similar analysis for the case when IPR are weak. Section 5 provides an empirical application of the licensing model to a case of electric bulb licensing in the United Kingdom. Finally, we conclude the paper with some remarks and suggestions for future research. An online appendix contains technical results.

2.2 Modelling Setup for Diffusion

The diffusion of new products has drawn considerable attention in marketing literature for both radical product innovations (e.g., Bass 1969; Mahajan et al. 1990, Sultan et al. 1990, Chandrasekaran and Tellis 2007) and incremental product innovations as "new generations" (e.g., Norton and Bass 1987, Mahajan and Muller 1996). A variety of extensions have incorporated competitive marketing mix variables to control the diffusion process (e.g., Robinson and Lakhani 1975, Horsky and Simon 1983, Kalish 1985, Horsky and Mate 1988, Bass et al. 1994, 2000, Krishnan et al. 1999). The diffusion literature deals mainly with monopolies of category level growth, but there are some extensions for rival brands (e.g., Parker and Gatignon 1994, Bayus et al. 2000, Prasad and Mahajan 2003, Savin and Terwiesch 2005, Libai et al. 2009).

In this section we consider a diffusion model for a non-durable product. The diffusion of a radical innovation follows a Bass-type specification driven by additional marketing mix variables. Similar to Gupta et al. (2006), we consider that $\dot{N}_t$ is the net customer growth, N_t are sales (instead of penetration in classical Bass model) generating returns $(p_t - c_t)\, N_t$, and there is a proportion $k \in (0,1)$ of defections and a potential level of customers $M > 0$. Managing the marketing mix the companies can control the dynamics of the diffusion-defections balance. Therefore, the customers' growth is given by

$$\dot{N}_t = \left[\left(a + u\frac{N_t}{M} \right) (M - N_t) - k\ N_t \right]\ W_t\left(A_t, p_t\right), \qquad N_0 = 0, \tag{2.1}$$

where $W\left(A_t, p_t\right)$ conveys the impact of advertising expenditure A_t and price p_t on the growth of sales. Multiplicative marketing mix impact has been previously considered in the diffusion literature, for example, by Bass et al. (1994). This model has been recently criticized by Fruchter and Van den Bulte (2011). We present an alternative approach that introduces some attractive features. First, we assume that advertising has a logarithmic impact on net sales diffusion, with an effect on market potentials and defections. Horsky and Simon (1983) suggested the use of advertising logarithms, but did not consider the effect on market potentials. Second, we consider that the impact of prices depends on the deviation from an ideal-point price $\bar{p} \geq 0$, and this benchmark evolves according to a reference inflation rate of consumption goods $r \geq 0$, so that the adoption process is faster when the distance $\left(e^{rt}\bar{p} - p_t\right)^2$ is smaller. Therefore we have specified the expression $W_t\left(A_t, p_t\right) = 1 + b \ln A_t - m\left(e^{rt}\bar{p} - p_t\right)^2$, and $a, b, m, u, M > 0$. Note that this specification allows $\bar{p} = 0$, so that $W\left(A_t, p_t\right)$ is monotonously decreasing with p_t. This might be the case of some mass consumption products, but for luxurious goods we would generally expect large values of $\bar{p}$.

Note that $W_t\left(A_t, p_t\right) = 1$ when $A_t = 1$ and the price equals to the ideal point $p_t = e^{rt}\bar{p}$. Then a stationary equilibrium is reached when $0 = \left(a + uN^*/M\right)\left(M - N^*\right) - kN^*$. If $u = 0$, with $a + k \neq 0$ the solution is $N^* = Ma/\left(a + k\right)$ (which tends to M when the defection parameter $k \downarrow 0$ decrease or $a \uparrow \infty$). For the general case, when $u \neq 0$, the long-term solution is:

$$N^* = M\frac{(u-a-k) \pm \sqrt{(u-a-k)^2 + 4au}}{2u},$$

which is smaller than the market potential. Obviously, when there are no defections, $k = 0$, the long-term equilibrium reaches the market potential as

$$N^* = (M/2u)\left((u-a) + \sqrt{(a+u)^2}\right) = M$$

We assume that the innovation is a variable-costs product with marginal cost $c_t = c_0\ e^{\pi t}$ where $c_0 > 0$, and π can be negative, zero, or positive, depending on whether the cost dynamics is dominated by industrial inflation or learning effects, or both are balanced. Note that in most models $c_t = c_0$, and we will stress this case. Denote by $i > 0$ the firm time-preference discount rate, that satisfies $i > \pi$ and $i > r$. The firm's present value of future profits is given by $\Pi = \int_0^\infty e^{-it}\left((p_t - c_t)\,N_t - A_t\right)\ dt$. In a monopolistic setting the firm faces the problem of maximizing profit Π subject to the diffusion equation (2.1).

Using this framework, we consider two competitive differential games. The first one describes a market where the IPR are strongly protected, and the second one describes a market with weak IPR protection and piracy.

2.3 Licensing Radical Innovations in Markets with Strong IPR

Denote by the letter h the firm (licensor or IPR holder) that holds a license in a market with an IPR protection. The IPR holder would be willing to license its innovation, if the additional revenue from licensing is positive and the monopoly's profits are not higher than those of oligopoly with competing licensees. Next, we describe the two possible scenarios: a monopolistic strategy versus licensing strategy.

Strategy 1 Holding a monopolistic position in the market. The firm faces the problem of maximizing profit by choosing price and advertising effort:

$$\boxed{\begin{array}{ll} \max \Pi^h_{mon}\left(p^h_t, A^h_t, N^h_t\right) = & \int_0^\infty e^{-it}\left(\left(p^h_t - c^h_t\right) N^h_t - A^h_t\right)\ dt \\ \text{s.t.} & \dot{N}^h_t = \left[\left(a + u\frac{N^h_t}{M}\right)\left(M - N^h_t\right) - k_h N^h_t\right] W_t\left(A^h_t, p^h_t\right), \\ & N^h_0 = 0, \end{array}}$$

where p^h is the product price and A^h the marketing effort, c^h_t are the unit costs, e^{-i} the discount parameter. Denote by Π^h_{mon} the optimal value of the monopolist.

Alternatively, the firm can consider licensing its innovation. Then, the sales diffusion of the IPR holder is driven by

$$\dot{N}^h_t = \left[\left(a + u\frac{N^h_t}{M} + g\frac{N^l_t}{M}\right)\left(M - N^h_t - N^l_t\right) - k_h N^h_t\right] W^h_t\left(A^h_t, A^l_t, p^h_t, p^l_t\right), \qquad (2.2)$$

with $W^h_t\left(A^h_t, A^l_t, p^h_t, p^l_t\right) = 1 + b\ln A^h_t + s\ln A^l_t - m\left(e^{rt}\overline{p} - p^h_t\right)^2 + d\left(e^{rt}\overline{p} - p^l_t\right)^2$, where $b, s, m, d > 0$.

Also, the IPR holder charges to each licensee a royalty fee over sales r^h_t and a fixed fee f^h_t. We consider a market potential of L licensees. In order to make the problem tractable, we consider that all licensed companies are relatively homogenous with constant marginal cost c^l_t, selling at the same price p^l_t. Therefore, we consider the aggregated sales of all licensees on a single brand l, and we assume that N^l_t are the sales of all licensees and A^l_t is the total marketing effort. The growth rate $\dot{N}^l_t$ depends also on the penetration of licensed companies l_t/L, defined as follows:

$$\dot{N}^l_t = \left[\left(\alpha + \upsilon\frac{N^h_t}{M} + \gamma\frac{N^l_t}{M} + \chi\frac{l_t}{L}\right)\left(M - N^h_t - N^l_t\right) - k_l N^l_t,\right] W^l_t\left(A^h_t, A^l_t, p^h_t, p^l_t\right), \quad (2.3)$$

with $W^l_t\left(A^h_t, A^l_t, p^h_t, p^l_t\right) = 1 + \beta\ln A^h_t + \sigma\ln A^l_t + \mu\left(e^{rt}\overline{p} - p^h_t\right)^2 - \delta\left(e^{rt}\overline{p} - p^l_t\right)^2$, where $\beta, \sigma, \mu, \delta > 0$.

The number of licensed firms, denoted by l_t, follows a Bass diffusion model in the following way:

$$\dot{l}_t = \left[\left(z_1 + z_2 \frac{l_t}{L} + z_3 \frac{N_t^l}{M} \right) (L - l_t) - z_4 l_t \right] \; L_t \left(f_t^h, r_t^h \right), \tag{2.4}$$

with $L_t \left(f_t^h, r_t^h \right) = 1 - z_5 \left(e^{rt} \overline{f} - f_t^h \right)^2 - z_6 \left(e^{rt} \overline{r} - r_t^h \right)^2$, where $z_5, z_6 > 0$. The adoption rate of the licensee companies $\dot{l}_t$ depends on number of previously adopted companies with coefficient z_2, as well as on the market penetration level of licensees' sales with coefficient z_3. In most of the scenarios we assume that there is no deflection of licensees, i.e. $z_4 = 0$. Additionally, the adoption rate of the licensees is controlled by the license fees f_t^h and r_t^h.

The IPR firm h and the licensee l sell their brands at different prices, and each firm benefits from the rival advertising to lesser extent, similarly to the model adopted from Gupta et al. (2006), Libai et al. (2009) and Savin and Terwiesch (2005). We consider $W_t^h \left(A_t^h, A_t^l, p_t^h, p_t^l \right)$ and $W_t^l \left(A_t^h, A_t^l, p_t^h, p_t^l \right)$ with positive parameters, therefore for both players we assume that sales growth increases with the advertising of any firm h and l, and sales growth decreases (increases) with an increase of own (competitors') price, i.e. h and l brands are substitutes. As firms generally address their advertising effort to their targeted segment by emphasizing their own product, and we assume that $b > s$ and $\sigma > \beta$; i.e., the effect of the own advertising in the sales is larger than the competitors' one. Similarly to Dockner and Jorgensen (1988), we assume that for price parameters $m > d$, $\delta > \mu$; i.e., the effect of the own price is larger than that of the competitors', which means that if all firms deviate from the ideal prices, they will encounter a decrease in their sales growth.

Denote by c_t^h, c_t^l the unit costs of the IPR holder and the licensee, respectively, which may be even identical if the production license covers all the know-how required for production. In this context, the following strategy is considered:

Strategy 2 Allowing a licensed substitute. Consider two substitutive brands (the patent holder h and the licensee l). The licensing strategy is characterized by a dynamic Nash equilibrium as follows:

- LICENSEE: Given the decisions of the licensor $\left\{A_t^h, p_t^h, r_t^h, f_t^h\right\}$, the licensees solve the problem:

$$\boxed{\begin{array}{ll} \max \Pi_{lic}^l\left(p_t^l, A_t^l, N_t^l\right) = & \int_0^\infty e^{-it}\left(\left(p_t^l - c_t^l - r_t^h\right) N_t^l - A_t^l - f_t^h l_t\right)\, dt \\ \text{s.t.} & (2.2),\ (2.3),\ (2.4) \end{array}}$$

and $N_0^l = 0$, $N_0^l = 0$, $l_0 = 0$.

- LICENSOR (IPR HOLDER): Given the decisions of the licensees $\left\{A_t^l, p_t^l\right\}$, the licensor solves the problem

$$\boxed{\begin{array}{ll} \max \Pi_{lic}^h\left(r_t^h, f_t^h, p_t^h, A_t^h, N_t^h\right) = & \int_0^\infty e^{-it}\left(\left(p_t^h - c_t^h\right) N_t^h - A_t^h + f_t^h l_t + r_t^h N_t^l\right)\, dt \\ \text{s.t.} & (2.2),\ (2.3),\ (2.4) \end{array}}$$

and $N_0^l = 0$, $N_0^l = 0$, $l_0 = 0$.

- In the licensing scenario, we denote by Π_{lic}^h and Π_{lic}^l the optimal profits for licensor and licensee companies, respectively, in an open-loop Nash equilibrium (for a definition see the Appendix).

The dynamic Nash equilibrium is generally defined using two alternative approaches: the *open-loop* Nash equilibrium and the *closed-loop* Nash equilibrium associated to different information structures. In an open-loop equilibrium, the decision of each agent satisfies the first order conditions of its maximization problem *ceteris paribus* the actions of the remainder players. By contrast, in a closed-loop Nash equilibrium it is assumed that each agent knows exactly how the other players will react to their decisions and anticipate these reactions in their first order conditions (see the appendix for a more formal description). Such managerial omniscience is generally unrealistic, but when it occurs the (closed-loop) equilibrium path is more robust to dynamic deviations, meaning that the closed-loop equilibrium is identified with a subgame-perfect equilibrium. In this paper we consider licensing solution with open-loop information structure.

2.3.1 Optimal Strategic Solution and Numerical Results

To determine whether the licensing strategy is implemented, we compute the optimal solution with and without licensing.

Licensing decision The decision to license is viable if and only if: $\Pi^h_{lic} \geq \Pi^h_{mon}$ and $\Pi^l_{lic} \geq 0$.

To solve the viability of licensing for a particular parametrization of the model, we should compute the first order conditions for each firm and study if the licensing conditions are verified. Propositions 1 and 2 in the online Appendix provide the first order conditions for the optimal policies based on Strategy 1 and Strategy 2, respectively. The solution is characterized by a Boundary Value Problem (BVP); i.e. a differential equation system with initial and terminal conditions. In order to discuss the optimality of the monopolistic approach (Strategy 1) compared with the licensing decision (Strategy 2) for the IPR holder, we should solve the optimal control systems substituting the optimal control expression in the associated BVP. The solution can be computed numerically solving the BVP with a Galerkin-Collocation method (for an introduction, see Esteban-Bravo and Vidal-Sanz, 2007).

We compute the optimal policies based on Strategy 1 and Strategy 2 for a set of parameters. As a base case, we assume an arbitrary total market size of 4000 units. We use coefficients of innovation of $a = 0.002$ for the licensor sales, $\alpha = 0.002$ for the licensee sales and the coefficients of imitation of $u = 0.2$; $g = 0.2$ for the licensor sales, and $\upsilon = 0.2$; $\gamma = 0.2$; $\chi = 0.02$ for the licensee sales. We assume that the market of potential licensees is $L = 90$, with a licensees coefficient of innovation $z_1 = 2$, a coefficient of imitation of $z_2 = 0.5$ and a sales impact coefficient $z_3 = 5$. The deflection rates for the three state equations are set to 0. We also assume that the variable cost is equal to

$c_0 = 20$ both for the licensor and licensees (and $\pi = 0$). We consider that the ideal-point of prices, royalty fees and fixed fees are $\bar{p} = 100$; $\bar{f} = 1200$; $\bar{r} = 10$; and this benchmark evolves according to an inflation rate $r = 0.07$. The sensitivity to the deviations from these ideal-points are set to $m = 0.0007$, $d = 0.0002$ for the licensor sales; $\mu = 0.0002$, $\delta = 0.0007$ for the licensees sales; and $z_5 = 0.0000015$; $z_6 = 0.00015$ for the licensees population. The efficiency of the advertising is set to $b = 0.01$; $s = 0.005$ for the licensor; and $\beta = 0.005$; $\sigma = 0.01$ for the licensees. We assume a discount rate of $i = 0.1$.

For this set of parameters, the optimal profit for monopolist is $\Pi^h_{mon} = 9.6860 \cdot 10^6$; and the optimal profits of the licensor and the licensee are $\Pi^h_{lic} = 1.0620 \cdot 10^7$ and $\Pi^l_{lic} = 6.3513 \cdot 10^5$, respectively. The results are not surprising. The value of licensing is $\Delta = \left(\Pi^h_{lic} - \Pi^h_{mon}\right) = 9.34 \cdot 10^5$. Indeed, there is a clear incentive for the IPR holder to license the innovation, because licensees pay with royalty fees. Actually, the discounted licensing revenue is about 50% of the total profit of the licensor (decreasing from an initial 90% down to around 40%). Figure 1 shows that the discounted profits of the IPR holder increase rapidly to a maximum, and then decay exponentially. But for the licensing strategy a higher value is achieved at a faster rate. The discounted profits of the licensee are smaller, but in the long term decay quite slowly.

Figure 1. Discounted optimal profits of the Monopolist, Strong IPR holder and Licensee

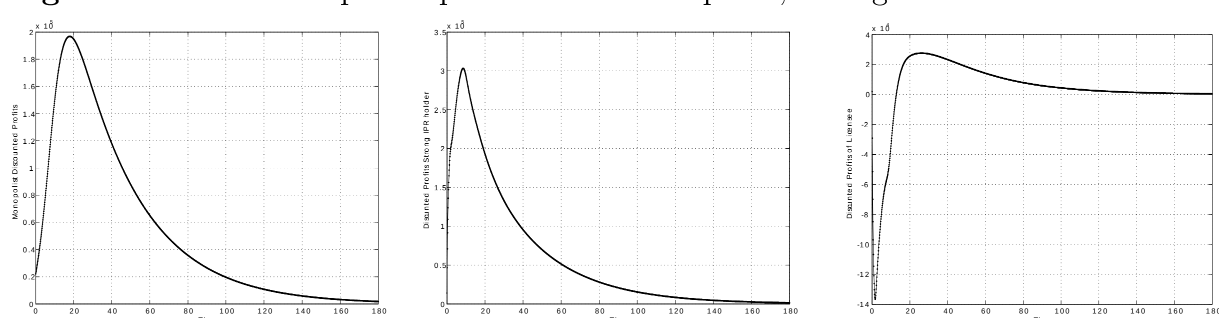
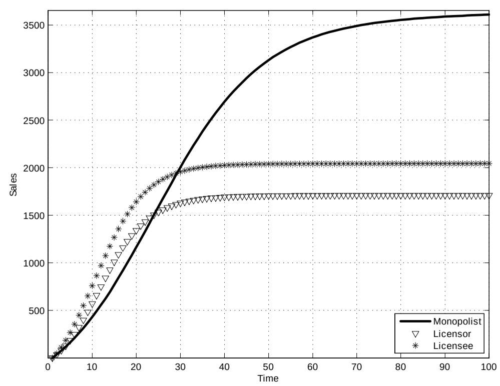


Figure 2 depicts how the sales diffusion is accelerated when the licensing strategy is implemented. Initially, the IPR holder has more sales when being a licensor than in a monopolistic position. The IPR holder implicitly also benefits from the advertising efforts and the cross word-of-mouth influence of the licensees, which leads to faster diffusion and, as a result, more sales per period. Besides, it also gains licensing revenues. For the IPR holder, taking the monopolistic strategy provides, after some point of time, a higher level of sales than those obtained if the licensing strategy is implemented. However, aggregated licensor and licensee sales dominate the monopolist sales before covering the market potential due to faster diffusion.

Figure 2. Sales diffusion in the context of strong IPR



Discounted optimal prices and discounted optimal marketing effort investments decay exponentially for all the agents (see Figure 3).

Figure 3. Optimal discounted prices and marketing effort under licensing

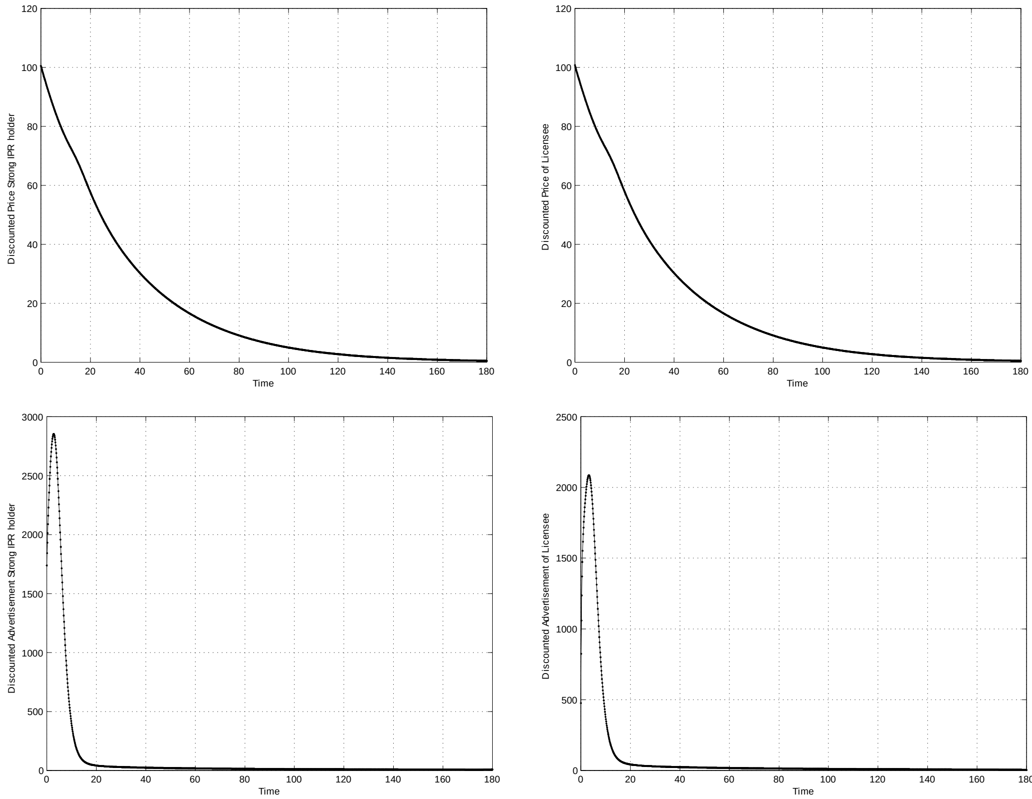
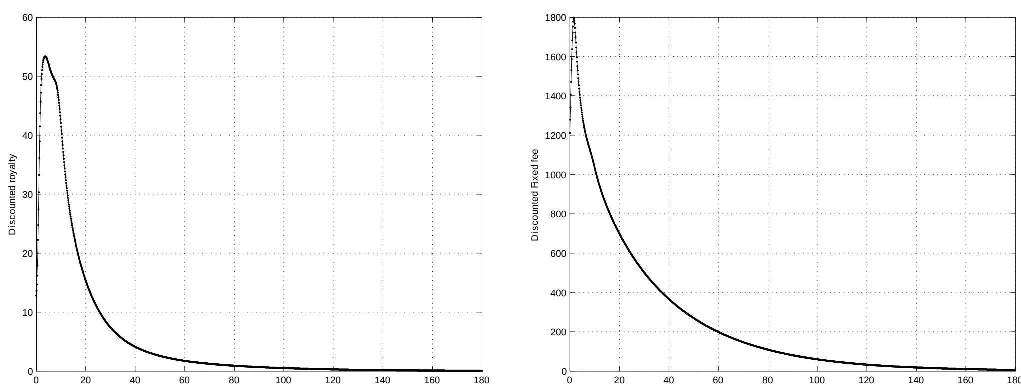


Figure 4 depicts the discounted licensing payments, both royalties and fixed fees. Both of them decay exponentially as the market becomes mature.

Figure 4. Discounted licensing payments



2.3.2 Comparative Statics

To get in-depth insights about the impact of the licensees, we provide a sensitivity analysis to the model's parameters. Consider all the parameters in a vector θ. Using envelope theorem for differential games, we compute how "sensitive" the cumulative profits $\Pi^h_{mon}, \Pi^h_{lic}, \Pi^l_{lic}$ are to changes in the value of the parameters in case of Strategy 1 and Strategy 2, respectively, and the partial derivatives $\partial\Delta(\theta_0)/\partial\theta$ (see the online Appendix for analytical details about the envelope theorem; the partial derivatives with respect to each model parameter are available from authors). These expressions involve some integrals that cannot be solved analytically. Thus, Table 1 reports the numerical values of the sensitivity analysis.

Table 1: Comparative statics for licensing and monopolistic strategies in strong IPR framework

θ	$\frac{\partial}{\partial\theta}\Pi^h_{lic}\,(\cdot 10^8)$	$\frac{\partial}{\partial\theta}\Pi^l_{lic}\,(\cdot 10^8)$	$\frac{\partial}{\partial\theta}\Pi^h_{mon}\,(\cdot 10^8)$	$\frac{\partial\Delta}{\partial\theta}\,(\cdot 10^8)$
a	0.4661	−0.3264	0.4681	−0.002
u	0.0746	−0.0616	0.1196	−0.045
M	0.000022	0.0000049	0.000027	−0.000005
m	−0.5842	−0.6066	−1.3098	0.7256
$\bar{p}$	0.00052	0.0006	0.00098	−0.00046
b	0.2521	−0.2131	0.1887	0.0634
k_h	−2.0407	2.0308	−0.6618	−1.3789
i	−3.834	−0.7721	−4.0179	0.1839
r	3.3085	0.9313	4.0380	−0.7295
s	0.2506	−0.1937	0	0.2506
d	−0.4060	−0.6755	0	−0.4060
g	0.0915	−0.0754	0	0.0915

θ	$\frac{\partial}{\partial\theta}\Pi_{lic}^{h}\,(\cdot 10^{8})$	$\frac{\partial}{\partial\theta}\Pi_{lic}^{l}\,(\cdot 10^{8})$	$\frac{\partial}{\partial\theta}\Pi_{mon}^{h}$	$\frac{\partial\Delta}{\partial\theta}\,(\cdot 10^{8})$
α	-0.1468	0.3205	0	-0.1468
υ	-0.0400	0.0593	0	-0.0400
γ	-0.0489	0.0726	0	-0.0489
μ	-0.4803	-0.3269	0	-0.4803
δ	-0.7131	-0.7767	0	-0.7131
β	-0.1473	0.2510	0	-0.1473
σ	-0.1497	0.2350	0	-0.1497
k_l	1.9515	-2.2347	0	1.9515
χ	-0.1407	0.2770	0	-0.1407

θ	$\frac{\partial}{\partial\theta}\Pi_{lic}^{h}\,(\cdot 10^{8})$	$\frac{\partial}{\partial\theta}\Pi_{lic}^{l}\,(\cdot 10^{8})$	$\frac{\partial}{\partial\theta}\Pi_{mon}^{h}$	$\frac{\partial\Delta}{\partial\theta}\,(\cdot 10^{8})$
z_1	0.00024	0.000069	0	0.00024
z_2	0.000088	0.000063	0	0.000088
z_3	0.0000001	-0.00004	0	0.0000001
z_4	-0.0064	0.0093	0	-0.0064
z_5	-1177.53	1316.51	0	-1177.53
z_6	-36.39	37.54	0	-36.35
$\bar{r}$	0.00062	-0.00062	0	0.00062
$\bar{f}$	0.00002	-0.00002	0	0.00002
L	0.0004	-0.0006	0	0.0004

The results are not surprising. We found negative values of $\partial\Delta/\partial a$ and $\partial\Delta/\partial u$ implying that the higher the innovation and word-of-mouth parameters are, the less the IPR holder is willing to license. When both a and u are large, product sales take off rapidly and the advantage of licensing to speed up the rate of diffusion of the innovation is not so clear. In addition, we found positive values of $\partial\Delta/\partial b$, $\partial\Delta/\partial s$ and $\partial\Delta/\partial g$ implying that the higher the impact of direct (or cross) advertising and the word-of-mouth on the licensor's sales growth, the more desirable the licensing strategy is.

An increase of licensor's price sensitivity m implies that licensing is more interesting since the monopoly profits are reduced more than those of licensing scenario. However, an increase of licensees price impact parameter d make licensing strategy less desirable for the IPR holder. We also found a negative value of $\partial\Delta / \partial\bar{p}$, indicating that for products with higher desired price level the IPR holder rather prefers to be in a monopolistic position, than to share the market with licensee companies. Furthermore, the faster the desired price level evolves (higher inflation rate r), the more a monopolistic position is preferred.

Interestingly, it does not appear to be worthwhile licensing with an increase of the IPR holder's deflection rate k_h, as the negative impact of higher deflection rate on monopoly profit is smaller than on licensor profit. Importantly, discounting parameter increments do improve the value of licensing as a takeoff anticipation becomes crucial. In other words, the more a firm is impatient to be rewarded for its innovation, the more an IPR holder is willing to license.

There is no a straightforward relationship between market potential M and licensing strategy. Computing the solution with different parameters, we found out that when cross advertising effects are moderate-to-large (higher than 0.005), then the higher market potential the more desirable a licensing strategy is. By contrast, when the cross advertising effects are smaller than 0.005, then the higher the market potential the less desirable a licensing strategy is. This strengthens the idea that companies may consider both, cross-benefits of advertising and the size of the market as structural elements to consider before a decision is made.

As it comes to the parameters related to the licensee sales growth, the parametric changes which speed up the licensee sales (higher a, β, γ, υ, χ, σ, and lower k_l) make licensing strategy less desirable. Intuitively, faster licensee sales imply that the IPR holder receives more royalty revenues earlier. However, it also implies that the licensees capture the market faster, leaving the IPR holder with less sales revenues.

Moreover, a higher licensee market potential L, larger z_1, z_2, z_3, and smaller z_4, z_5, z_6 (speeding up of licensee companies diffusion and reducing the sensitivity to fees), make licensing a more attractive strategy. This is because if the IPR holder chooses the licensing strategy, he would prefer that the licensees "diffuse" faster so that he gets licensing fees earlier. These results are relatively stable, although we have noticed that for high levels of production costs $\partial\Pi^h_{lic}/\partial z_3$ becomes small and negative, but if cross advertising effects is simultaneously set to 0.003 the effect is again positive.

It is important to note that most of the parametric changes that make licensing contract more attractive for the licensor decrease the total profits earned by the licensees. However, as long as the licensees get positive profits, licensing is an attractive strategy for the licensees, as the licensees would get zero profits by not accepting the licensing contract.

2.4 The Case of Weak IPR Markets

Product licensing decisions should be reconsidered when unlicensed pirate imitations can be “commercialized”. The IPR holder faces a weak type of competition from a substitutive product with worse characteristics. But piracy also speeds up the product diffusion, bringing some issues that were not present in monopoly. In this context it is not so crucial to allow rival pirates to use legal licenses given that the license must be cheap enough to engage them into the legal binding.

In the weak IPR context, we can study the sales of three possible products: (1) the IPR holder sales N^h_t; (2) the sales of copies produced by the firms without IPR N^c_t;and (3) the sales of licensed product N^l_t if a license agreement is signed. We consider that only products (1) and (2), or products (1) and (3) are simultaneously observed in the market. We say that firms that only commercialize products of type (2) follow a pirating strategy. Often, the quality of the original and copied product is usually different, and therefore the competition of IPR holder firm and pirates is weaker than in the case studied in

Section 2.3. Therefore, we relax the assumption of a joint market potential and assume that brands can develop independently, with a market potential of M^h for products of type (1) and M^c for products of type (2), so that $M = M^h + M^c$, and separated diffusions (similarly to Parker and Gatignon, 1994, and Gupta et al., 2006)).

The sales of pirate products N_t^c depend on the number of pirates l_t^c, and the dynamics of both variables are cross-related. Therefore, if the licensee is not granted, the diffusion of the innovative and pirate products and pirate companies are described by the following equations:

$$\dot{N}_t^h = \left[\left(a + u\frac{N_t^h}{M^h} + g\frac{N_t^c}{M^c} \right) \left(M^h - N_t^h \right) - k_h N_t^h \right] \; W_t^h \left(A_t^h, A_t^c, p_t^h, p_t^c \right) \tag{2.5}$$
$$W_t^h \left(A_t^h, A_t^c, p_t^h, p_t^c \right) = 1 + b \ln A_t^h + s \ln A_t^c - m \left(e^{rt}\overline{p} - p_t^h \right)^2 + d \left(e^{rt}\overline{p^c} - p_t^c \right)^2$$

$$\dot{N}_t^c = \left[\left(\alpha + \upsilon\frac{N_t^h}{M^h} + \gamma\frac{N_t^c}{M^c} + \chi\frac{l_t^c}{L} \right) \left(M^c - N_t^c \right) - k_c N_t^c \right] \; W_t^c \left(A_t^h, A_t^c, p_t^h, p_t^c \right) \tag{2.6}$$
$$W_t^c \left(A_t^h, A_t^c, p_t^h, p_t^c \right) = 1 + \beta \ln A_t^h + \sigma \ln A_t^c - \delta \left(e^{rt}\overline{p^c} - p_t^c \right)^2 + \mu \left(e^{rt}\overline{p} - p_t^h \right)^2$$

and

$$\dot{l}_t^c = \left[\left(z_1 + z_2\frac{l_t^c}{L} + z_3\frac{N_t^c}{M^c} \right) \left(L - l_t^c \right) - z_4 l_t^c \right] \tag{2.7}$$

By contrast, we assume that in the licensing scenario, products have relatively homogeneous quality, and therefore the demand structure is entirely analogous to model (2.2), (2.3), (2.4), with market potential $M = M^h + M^c$, and the sales are distributed among both firms accordingly to this model.

Typically the diffusion parameters of the IPR firm product N_t^h and the licensed product N_t^l are higher than in the copy N_t^c, due to higher quality of the innovation that increases the market potential and the speed of the diffusion (i.e., $a > \alpha$, $u > \upsilon$, $g > \gamma$), also $b > \sigma$; $s > \beta$.

The licensor considers two possible scenarios: competition with pirating strategy versus licensing strategy (Strategy 2) with $M = M^h + M^c$.

Strategy 3 Unlicensing benchmark scenario with piracy. The equilibrium is characterized by the dynamic Nash equilibrium as follows:

- Given the decisions of the IPR holder $\left\{A_t^h, p_t^h\right\}$ the pirates, solve

$$\boxed{\begin{array}{ll} \max \Pi_{piracy}^c \left(p_t^c, A_t^c, N_t^c\right) = & \int_0^\infty e^{-it} \left(\left(p_t^c - c_t^c\right) N_t^c - A_t^c\right) \; dt \\ s.t. & eq. \; (2.5), (2.6), (2.7) \end{array}}$$

  with $N_0^c = 0$, $N_0^h = 0$.

- Given the decisions of the pirates $\{A_t^c, p_t^c\}$, the IPR holder solves the problem

$$\boxed{\begin{array}{ll} \max \Pi_{piracy}^h \left(p_t^h, A_t^h, N_t^h\right) = & \int_0^\infty e^{-it} \left(\left(p_t^h - c_t^h\right) N_t^h - A_t^h\right) \; dt \\ s.t. & eq. \; (2.5), (2.6), (2.7) \end{array}}$$

  with $N_0^c = 0$, $N_0^h = 0$.

- Under piracy, we denote by Π_{piracy}^h and Π_{piracy}^c the optimal profits of IPR holder and pirates, respectively obtained in the open-loop Nash equilibrium.

Alternatively, a licensing agreement can be signed, and therefore, the IPR holder and the licensee follow Strategy 2 defined in Section 2.3, rendering profits Π_{lic}^h, Π_{lic}^l.

2.4.1 Optimal Strategic Solution and Numerical Results

To determine whether the licensing strategy is implemented, consider the following decision rule:

Licensing decision The license decision is a viable equilibrium if and only if two conditions holds:

i) The IPR licensor gets higher profits with licensing than without licensing; i.e., $\Pi_{lic}^h \geq \Pi_{piracy}^h$,

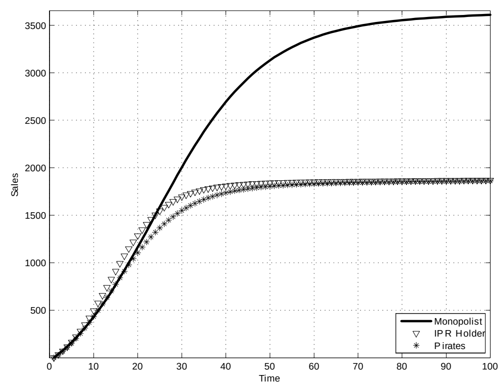
ii) The licensee obtains higher profits commercializing licensed products than unlicensed substitutes; i.e., $\Pi^l_{lic} \geq \Pi^c_{piracy}$.

In order to determine whether the Licensing agreement is viable under piracy, we need to compute the optimal solution based on Strategy 3. The first order conditions are given in Proposition 3 in the online Appendix. The solution is characterized by a BVP, which can be solved numerically.

We compute the optimal policies in a weak IPR context for a set of parameters. As a base case, we assume the set of parameters given Section 2.3, arbitrarily decomposing the market size $M = 4000$ in 2000 units for illegal copies and 2000 for licensed products. We also tested other asymmetric decompositions without a strong impact on the results (in particular, the impact of small changes is discussed in the comparative statics section). We assume lower quality of the innovation for the pirates, and thus, slower growth of pirate product sales. For that reason, we consider $\alpha = 0.001$, $\beta = 0.00005$, $\sigma = 0.0001$, $\gamma = 0.15$, $\upsilon = 0.15$, which are lower than the parameters considered in Section 2.3. For this set of parameters, in the context of the weak IPR protection with pirates, the optimal profits of the IPR holder and the pirates are $\Pi^h_{piracy} = 5.6249{\cdot}10^6$ and $\Pi^c_{piracy} = 2.7518{\cdot}10^6$, respectively. In the context of the weak IPR protection with licenses, the optimal profits of the IPR holder and the licensee are $\Pi^h_{lic} = 1.0620 \cdot 10^7$ and $\Pi^l_{lic} = 6.3513 \cdot 10^5$, respectively. The result suggests that the licensing strategy might be interesting for the IPR holder ($\Pi^h_{lic} > \Pi^h_{piracy}$), but not for the pirates ($\Pi^l_{lic} < \Pi^c_{piracy}$). As a consequence the license agreement is not implemented. Furthermore, recall that in a strong IPR context with analogous parameters, the optimal profit for monopolist is $\Pi^h_{mon} = 9.6860 \cdot 10^6$, so the difference of $\left(\Pi^h_{mon} - \Pi^h_{piracy}\right) = 4.\,0611 \cdot 10^6$ is the financial cost of piracy for the IPR holder, i.e. the economic loss caused to the innovator because of a weak IPR regulation. The dynamics of the discounted profits in the equilibrium is similar for both the IPR holder and the pirates; i.e., a fast growth followed by an exponential decay to zero. At the maximum pirates achieve roughly 3 times the maximum level of discounted profits obtained by a licensee, but decay faster.

The previous result also suggests that although piracy may speed up the product diffusion of the IPR holder through word of mouth and communication of pirates (Figure 5 shows the sales diffusion in a market with pirates compared to that of a monopolist IPR holder), this advantage is not enough to compensate the innovator for the market-share loss. Under a licensing agreement, the royalties would reduce the damage, rendering $\Pi^h_{lic} = 1.0620 \cdot 10^7$, which is higher than the return in a monopoly. However, pirates do not receive reciprocal benefits from the licensing contract.

Figure 5: Sales diffusion in the context of weak IPR.



The dynamic pattern of the optimal discounted prices and marketing effort investments for weak IPR protection scenario is similar that of an analogous licensor in a strongly protected market. By contrast, pirates invest in advertising roughly 1/8 of the IPR holder, and charge approximately half of its price.

2.4.2 Comparative Statics

In this section we study the effect of parametric changes on the returns drawn from the different strategies in a weak IPR framework. Using the envelope theorem for differential games, we compute the impact of parameter change on the profits Π^h_{lic}, Π^l_{lic}, Π^h_{piracy}, Π^c_{piracy} and the relative advantage of licensing for IPR holder $\Delta^h = \left(\Pi^h_{lic} - \Pi^h_{piracy}\right)$ and for pirates $\Delta^l = \left(\Pi^l_{lic} - \Pi^c_{piracy}\right)$. Table 2 shows numerical values of partial derivatives of the profit functions, $\partial\Delta^h(\theta_0)/\partial\theta_i$, and $\partial\Delta^l(\theta_0)/\partial\theta_i$ for the different parameters in the model; that is how "sensitive" the cumulative profits are to changes in the value of the parameters of the model considering a licensed and unlicensed market.

Table 2: Comparative statics for licensing and piracy strategies in the weak IPR framework

θ	$\frac{\partial\Pi^h_{lic}}{\partial\theta}(\cdot 10^8)$	$\frac{\partial\Pi^l_{lic}}{\partial\theta}(\cdot 10^8)$	$\frac{\partial\Pi^h_{piracy}}{\partial\theta}(\cdot 10^8)$	$\frac{\partial\Pi^c_{piracy}}{\partial\theta}(\cdot 10^8)$	$\frac{\partial\Delta^h}{\partial\theta}(\cdot 10^8)$	$\frac{\partial\Delta^l}{\partial\theta}(\cdot 10^8)$
a	0.4661	-0.3264	0.1048	0.0444	0.3613	-0.3708
u	0.0746	-0.0616	0.0275	0.0094	0.0471	-0.071
M	0.000022	0.0000049				
M^h			0.000085	-0.0062	-0.000063	0.0062049
M^c			-0.000018	0.000042	0.00004	-0.0000371
m	-0.5842	-0.6066	1.9415	-0.6619	-2.5257	0.0553
$\bar{p}$	0.00052	0.0006	-0.0527	0	0.0532	0.0006
$\bar{p}^c$	0	0	0.0244	0.00054	-0.0244	-0.00054
b	0.2521	-0.2131	0.0775	0.0311	0.1746	-0.2442
k_h	-2.0407	2.0308	-0.4457	-0.1419	-1.595	2.1727
i	-3.834	-0.7721	-2.1205	-1.0592	-1.7135	0.2871
r	3.3085	0.9313	-218.26	1.0451	221.5685	-0.1138
s	0.2506	-0.1937	0.0691	0.0213	0.1815	-0.215
d	-0.4060	-0.6755	2.0384	0.4557	-2.4444	-1.1312
g	0.0915	-0.0754	0.0253	0.0087	0.0622	-0.0841

θ	$\frac{\partial \Pi^h_{lic}}{\partial \theta}$ $(\cdot 10^8)$	$\frac{\partial \Pi^l_{lic}}{\partial \theta}$ $(\cdot 10^8)$	$\frac{\partial \Pi^h_{piracy}}{\partial \theta}$ $(\cdot 10^8)$	$\frac{\partial \Pi^c_{piracy}}{\partial \theta}$ $(\cdot 10^8)$	$\frac{\partial \Delta^h}{\partial \theta}$ $(\cdot 10^8)$	$\frac{\partial \Delta l}{\partial \theta}$ $(\cdot 10^8)$
α	-0.1468	0.3205	0.0478	0.0666	-0.1946	0.2539
υ	-0.0400	0.0593	0.0091	0.0202	-0.0491	0.0391
γ	-0.0489	0.0726	0.0083	0.0185	-0.0572	0.0541
μ	-0.4803	-0.3269	0.3613	1.9359	-0.8416	-2.2628
δ	-0.7131	-0.7767	12.9447	-1.8911	-13.6578	1.1144
β	-0.1473	0.2510	0.0281	0.0584	-0.1754	0.1926
σ	-0.1497	0.2350	0.0468	0.0426	-0.1965	0.1924
k_l	1.9515	-2.2347	-0.0387	-0.2753	1.9902	-1.9594
χ	-0.1407	0.2770	0.0381	0.0567	-0.1788	0.2203

θ	$\frac{\partial}{\partial \theta}\Pi^h_{lic}$ $(\cdot 10^8)$	$\frac{\partial \Pi^l_{lic}}{\partial \theta}$ $(\cdot 10^8)$	$\frac{\partial \Pi^h_{piracy}}{\partial \theta}$ $(\cdot 10^8)$	$\frac{\partial \Pi^c_{piracy}}{\partial \theta}$ $(\cdot 10^8)$	$\frac{\partial \Delta^h}{\partial \theta}$ $(\cdot 10^8)$	$\frac{\partial \Delta^l}{\partial \theta}$ $(\cdot 10^8)$
z_1	0.00024	0.000069	0.00013	0.00013	0.00011	-0.000061
z_2	0.000088	0.000063	0.000079	0.000082	0.000009	-0.000019
z_3	0.000001	-0.00004	0.000041	0.000041	-0.00004	-0.000081
z_4	-0.0064	0.0093	-0.0031	-0.0031	-0.0033	0.0124
z_5	-1177.53	1316.51	0	0	-1177.53	1316.51
z_6	-36.39	37.54	0	0	-36.39	37.54
$\bar{r}$	0.00062	-0.00062	0	0	0.00062	-0.00062
$\bar{f}$	0.00002	-0.00002	0	0	0.00002	-0.00002
L	0.0004	-0.0006	0.00023	0.00023	0.00017	-0.00083

In the context of weak IPR markets, licensing is the recommendable strategy for the IPR holder because an increase of the pirates diffusion does not imply an increase of its profits (as pirates do not pay licensing fees). Similar to the case of strong IPR markets, increasing the speed of licensee companies diffusion (larger z_1,z_2,z_3, and smaller z_4,z_5,z_6,) and the market potential of licensees L makes licensing more attractive strategy.

If we keep M constant, and increase M^h whereas decreasing M^c we can consider the marginal change in the licensing advantage for IPR holder $\left(\partial\Delta^h/\partial M^h - \partial\Delta^h/\partial M^c\right) = -1.03 \times 10^{-4} < 0$, and also the marginal change in the licensing advantage for pirates $\left(\partial\Delta^l/\partial M^h - \partial\Delta^l/\partial M^c\right) = 6.242 \times 10^{-3} > 0$. When the IPR increases its share of the total market potential with respect to the pirates, then licensing is less (more) attractive for the licensor (pirates).

Table 2 also shows that $\partial\Delta^h/\partial\theta$ and $\partial\Delta^l/\partial\theta$ generally take values of opposite signs with the exception of those related to prices. This result implies that the more desirable licensing is for the IPR holder, the less willing the pirates are to accept the licensing contract. And vice versa - when a parametric change makes licensing more attractive for the pirates, the IPR holder is less willing to offer a licensing contract. Although we do not completely rule out the possibility of licensing contract viability, these results show that obviously licensing event is less common in weak IPR framework. This finding goes in line with empirical evidence by Arora and Ceccagnoli (2006), Gambardella et al. (2007), and Anand and Khanna (2000).

Previous research suggests that piracy can be beneficial for an IPR holder, since pirates accelerate diffusion via word-of-mouth (see Givon et al. 1995, and Givon et al. 1997). We also found some interesting insights studying the financial loss of the IPR holder due to piracy, that is the difference between the partial derivatives of profits of the IPR holder in a monopolistic scenario (see $\partial\Pi^h_{mon}/\partial\theta$ in Table 1) and those when pirates are present (see $\partial\Pi^h_{piracy}/\partial\theta$ in Table 2). The lower the coefficient of innovation (parameter a) is, the lower the IPR holder's financial loss is caused by the piracy. In other words, when the innovation parameter of the IPR holder is very small, the IPR holder could be better off tolerating some piracy. This goes in line with the results of Prasad and Mahajan (2003), who find that the lower the coefficient of innovation, the higher is the optimal tolerance level of piracy. Also, the higher the word of mouth parameter u is, the more willing the IPR holder is to be in a monopolist position. This is because for a high u the IPR holder has sufficient word of mouth and it is less willing to tolerate pirates (Prasad and Mahajan, 2003, find that piracy toleration is lower when word of mouth coefficient is either very small or very large). Prasad and Mahajan (2003) also find a minor effect of discount rate on piracy tolerance. By contrast, our results suggest that the higher the discount rate i is, the smaller the IPR holder's financial loss is due to the piracy. This result is reasonable, since when the discount rate is higher, faster diffusion of sales becomes more crucial for the IPR holder. Thus, the IPR holder may even prefer pirates to have some part of the total market potential, given that pirates speed up the diffusion.

2.5 Empirical Application

This section presents an empirical application of the proposed licensing model to incandescent light bulbs industry in the United Kingdom (UK). Light bulb industry emerged in the end of 19th century. After a number of years cooperating through prices and patent licensing, in 1925 the world leading lamp manufacturers negotiated the General Patent and Business Development Agreement (also known as the "Phoebus Agreement" after the administrative office—S.A. Phoebus— located in Geneva) originally set up to 1955, but broken to an end by the second world war. The Phoebus Agreement divided the world markets, giving to each party the right to the same annual proportion of the total business in each territory as they had in 1924, and set product standards. The Electric Lamp Manufacturers' Association (E.L.M.A.) got the control for lamp manufacturing in the UK. (E.L.M.A. included British Thomson-Houston Co. Ltd., Edison Swan Electric Co. Ltd., Metropolitan-Vickers Electrical Co. Ltd., General Electric Co. Ltd., Philips Electrical Ltd., Stella Lamp Co. Ltd., Cryselco Ltd., Siemens Electric Lamps & Supplies Ltd., Crompton Parkinson Ltd., British Electric Lamps Ltd., and Aurora Lamps Ltd.)

In 1937 the monopolist consortium E.L.M.A granted a non-exclusive non-transferable license to produce and sell electric lamps in the UK to British Luma Co-operative Electric

Lamp Society Ltd. (henceforth British Luma), a cooperative of small UK manufacturers. The output sold by British Luma was slightly larger than the sum of productions of the two smallest companies in E.L.M.A. In this section we study the E.L.M.A. and British Luma agreement using the licensing model to obtain structural explanations for their decision and compute the optimal patterns for their profits and sales. We use sales data from the UK Monopolies Commission Report on the Supply of Electric Lamps. The yearly lamp production data extends from 1924 till 1950. The licensing contract also specifies the number of lamp units that the Licensee is permitted to sell or otherwise dispose of for the duration of the License from 1937 till 1947. The difference between total lamp production and Licensee's sales is recorded as the Licensor annual sales. We use linear interpolation to obtain data for the missing observations. For this particular case, we consider that the decision problems faced by Licensor and the Licensee at the year of 1937 are given by the following structure:

- The licensee British Luma solves the problem

$$\boxed{\begin{array}{ll} \max \Pi^l_{lic} = & \int_0^\infty e^{-it} \left(\left(p^l_t - c\right)(1-r)\, N^l_t \right) \; dt \\ \text{s.t.} & (2.8),\ (2.9), \end{array}}$$

- The licensor E.L.M.A. solves the problem

$$\boxed{\begin{array}{ll} \max \Pi^h_{lic} = & \int_0^\infty e^{-it} \left(\left(p^h_t - c\right) N^h_t + \left(p^l_t - c\right) r N^l_t \right) \; dt \\ \text{s.t.} & (2.8),\ (2.9) \end{array}}$$

where

$$\dot{N}^h_t = \left[\left(a + u \left(\frac{N^h_t}{M^h} + \frac{N^l_t}{M^l} \right) \right) \left(M^h - N^h_t\right) \right] \left[1 - m \left(e^{rt}\overline{p} - p^h_t\right)^2 + d \left(e^{rt}\overline{p} - p^l_t\right)^2 \right], \tag{2.8}$$

and

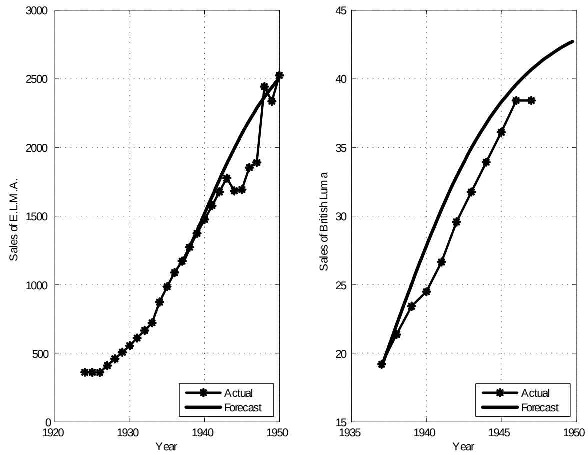
$$\dot{N}^{l}_{t} = \left[\left(\alpha + \upsilon \left(\frac{N_t^h}{M^h} + \frac{N_t^l}{M^l} \right) \right) \left(M^l - N_t^l \right) \right] \left[1 + \mu \left(e^{rt}\overline{p} - p_t^h \right)^2 - \delta \left(e^{rt}\overline{p} - p_t^l \right)^2 \right] \tag{2.9}$$

and $N_0^h = N_{1937}$, $N_0^l = 0$.

Note that, by the licensing contract, the Licensee pays a 3% royalty of the net selling price to the licensor, i.e. $r = 0.03$. By the contract the licensees should not supply lamps which are different in size characteristics from those sold by the Licensor. Thus, we assume that the word of mouth effects of licensor and licensee sales are the same. Furthermore, the growth equation for licensees number is absent from the system of state equations as there is only one licensee company throughout the licensing period. We did not have data on advertising, and therefore it was not included in the model. Two different market potentials were considered, similar to the weak IPR framework, as the agreement restricted the market potential for British Luma.

Regarding of the pricing of the electrical bulbs, by the licensing contract, British Luma does not take decision over the prices. By contrast, we consider how price competition affects the sales diffusion paths. We set the market potentials as $M^h = 3000 \bullet 10^5$ and $M^l = 40 \bullet 10^5$, and obtain the least squares estimators of the coefficients of innovation and imitation: $a = 0.012$, $u = 0.061$, $\alpha = -0.15$, $\upsilon = 0.32$, $m = 3.3$, $\delta = 24.15$, with $d = 1$ and $\mu = 10$. Using the estimated parameter values, a discount factor $i = 0.08$, an average measure of desirable price $\bar{p} = 0.05£$ for a unit of bulb, and a unit cost of $0.03£$, we solve numerically the dynamic game between the Licensor and the Licensee. The optimal discounted lifetime profits for E.L.M.A and British Luma are $1.26 \bullet 10^8 £$ and $8.45 \bullet 10^5 £$, respectively. Figure 6 depicts the optimal sales path for both companies over the considered time period, as well as the empirical data. Note that between 1939 and 1945 the second world war damaged European lamp sales even for the posterior years, and Phoebus Agreement was ended by that time.

Figure 6. Sales of E.L.M.A. and British Luma



A manager of a company which faces a decision to license (or alternatively, to become a licensee) could perform a similar analysis to get a measure of total lifetime profits from licensing and optimal product sales path. This would help to make licensing decisions. Also, by possessing external measures of sales sensitivity to marketing mix variables the manager could use the model to deduce optimal marketing mix decisions. Importantly, the analysis takes into account the word of mouth acceleration effect of the licensee sales on sales growth of the licensor and vice versa.

2.6 Concluding Remarks

This paper considers the use of licenses as a strategy to speed up the sales diffusion process of new products, commercializing the innovation and simultaneously licensing the product technology to one or more other firms. Our findings show that licensing is a beneficial strategy for the innovator who renounces to monopolistic power derived from the exclusivity, but draws cash-flow from royalties and exploits the advertising investment and positive word-of-mouth effects by licensees. The license agreement is feasible in markets with strong IPR, as licensor and licensee are benefited from the agreement. We compare this result with a weak IPR context where piracy speeds up the product diffusion but this does not compensate the IPR holder for the sales loss effect who would prefer to license in order to get some royalties. However, pirates do not generally find the licensing agreement interesting.

Managerial Implications: Radical product innovations considerably vary in their take-off times and the speed of diffusion. Most managers favor rapid acceleration of sales diffusion in order to receive quicker returns. As far as we know, this is the first paper that analyzes licensing as a strategy to accelerate the diffusion of radical product innovation in different IPR frameworks. We showed that licensing would provide considerable benefits to a firm for a certain set of parameter values, and analyzed the robustness of this property using comparative statics. Thus, a manager could conduct a similar analysis to decide for the benefits of product licensing and to have optimal marketing mix and licensing decisions over time. The empirical application illustrates how the proposed model can be adapted to different data sets. Note also that the model for weak IPR could be applied to explain the viability of some franchise agreements.

Limitation and Future Research: The proposed modeling approach leaves open many possibilities for future research. First, in the dynamic game solution we assume open-loop information structure, but there are other alternatives. Future work could extend the analysis to industries with more information using a closed loop or a Stackelberg equilibrium, although the computational task to solve these models is formidable. Second, we assume a model specification that builds upon the new product diffusion literature. Although the specification is well grounded, we consider that empirical testing of the model would be important. Unfortunately, the richness of the required data makes it somehow difficult, particularly for pirates in the case of weak IPR. To respond this type of challenge, researchers have suggested to use experimental designs as well as computer based simulations (e.g. Montaguti et al. 2002). We have mostly followed the second strategy. The comparative static analysis yields several hypotheses, some of which are testable. We leave this for further research.

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2.8 Appendix

Consider N players each of them setting a set of control variables u_{it}, a deterministic dynamic system $\dot{x}_t = g(x_t, u_t, t)$, $x(0) = x_0$ with $u_t = (u_{1t}, ..., u_{Nt})$. In a standard differential game, each player maximizes the profits $\Pi^i = \int_0^\infty G^i(u_t, x_t, t)\, dt$ subject to a dynamic system constraints, where G^i and g are continuously differentiable functions. In an (open-loop) Nash equilibrium (x^*, u^*)

$$\Pi^i(x^*, u_1^*, ..., u_N^*) = \max_{u_i, x} \Pi^i\left(x, u_1^*, ..., u_{i-1}^*, u_i, u_{i+1}^*, ..., u_N^*\right), \tag{2.10}$$

subject to the dynamic system constraint, for $i = 1, ..., N$. The open-loop Nash equilibrium $\{x^*, u_1^*, ..., u_N^*\}$ satisfies the first order conditions:

$$\frac{\partial H^i}{\partial u_{it}} = 0,\ \frac{\partial H^i}{\partial \lambda_{it}} = \dot{x}_t,\ \dot{\lambda}_{it} = -\frac{\partial H^i}{\partial x_t}, \tag{2.11}$$

for $i = 1, ..., N$, where $H^i(x_t, u_{1t}, ...u_{Nt}, t) = G^i(x_t, u_{1t}, ...u_{Nt}, t) + \lambda_{it}\, g(x_t, u_{1t}, ...u_{Nt}, t)$ is the Hamiltonian of player i. By contrast, the closed-loop Nash (i.e. feedback) equilibrium (x^{**}, u^{**}) is defined when (2.10) conditions are redefined in terms of the optimal response by other players to the state of the system, i.e.

$$\Pi^i(x^{**}, u_1^{**}, ..., u_N^{**}) = \max_{u_i, x} \Pi^i\left(x, u_1^{**}(x), ..., u_{i-1}^{**}(x), u_i, u_{i+1}^{**}(x), ..., u_N^{**}(x)\right),$$

subject to the dynamic system constraint, for $i = 1, ..., N$. The closed-loop Nash equilibrium satisfies the HJB first order conditions:

$$\frac{\partial H^i}{\partial u_{it}} = 0,\ \frac{\partial H^i}{\partial \lambda_{it}} = \dot{x}_t,\ \dot{\lambda}_{it} = -\frac{\partial H^i}{\partial x_t} - \sum_{j \neq i} \frac{\partial H^i}{\partial u_{jt}} \frac{\partial u_{jt}}{\partial x_t}, \tag{2.12}$$

for $i = 1, ..., N$, where the terms $\{\partial u_{jt}/\partial x_t\}_{j \neq i}$ means that the $i-th$ player anticipates the competitors' reaction to changes in the state variable when the player sets its policy, and these reactions are respectively weighted by their impact in the i-th player Hamiltonian $\partial H^i/\partial u_{jt}$. When $\partial u_{jt}/\partial x_t = 0$ or $\partial H^i/\partial u_{jt} = 0$ for all pairs $i \neq j$, then the open-loop and the closed-loop equilibrium are identical. Once the solution is computed we can evaluate the optimal values Π^{*i} for each player at equilibrium. Finally, a Stackelberg differential game equilibrium assumes asymmetric information, such that informed leaders behave as in the closed-loop equilibrium, and the ignorant followers behave as in an open-loop equilibrium. For a detailed introduction to optimal control and differential games, see e.g. Seierstad and Sydsaeter (1987).

Regarding the comparative statics, let us assume that the state and profit functions depend upon certain parameters $\theta \in \Theta \subset \mathbb{R}^K$ where Θ is an open set, so that $g_\theta(x_t, u_t, t)$ and $G^i_\theta(u_t, x_t, t)$ are smooth functions on Θ. Then the envelope theorem for differential games states that,

$$\left.\frac{\partial \Pi^{*i}_\theta}{\partial \theta}\right|_{\theta=\theta_0} = \int_0^\infty \left.\left[\frac{\partial H^i(x_t, u_t, t)}{\partial \theta} + \sum_{j \neq i} \frac{\partial H^i(x_t, u_t, t)}{\partial u_{jt}} \frac{\partial u_{jt}}{\partial \theta}\right]\right|_{\theta=\theta_0} dt$$

and this result is valid for both, open-loop and closed-loop equilibrium. A simple proof can be found, e.g., in LaFrance et al. (1991) for the case when $N = 1$, and in Caputo (2007) for differential games.

Next we provide the Hamilton-Jacobi-Bellman (HJB) first order conditions for the optimal policies based on Strategy 1 and Strategy 2 in markets with strong IPR.

Proposition 1 *The optimal pricing and advertising decisions for a monopolistic IPR holder (Strategy 1) are given by:*

$$\begin{aligned} p_t^h &= \quad e^{rt}\bar{p} + \frac{e^{-it}N_t^h}{\lambda_t^h 2m\left[\left(a+u\frac{N_t^h}{M}\right)\left(M-N_t^h\right)-k_h N_t^h\right]}, \\ A_t^h &= \quad be^{it}\lambda_t^h\left[\left(a+u\frac{N_t^h}{M}\right)\left(M-N_t^h\right)-k_h N_t^h\right], \end{aligned}$$

where N_t^h, λ_t^h are the solution to the Boundary Value Problem (BVP) defined by

$$\begin{aligned}
\dot{N}_t^h = & \left[\left(a + u\tfrac{N_t^h}{M}\right)\left(M - N_t^h\right) - k_h N_t^h\right]\left(1 + b\ln A_t^h - m\left(e^{rt}\bar{p} - p_t^h\right)^2\right) \\
\dot{\lambda}_t^h = & -e^{-it}\left(p_t^h - c_t^h\right) \\
& -\lambda_t^h\left(\tfrac{u}{M}\left(M - N_t^h\right) - \left(a + u\tfrac{N_t^h}{M}\right) - k_h\right)\left(1 + b\ln A_t^h - m\left(e^{it}\bar{p} - p_t^h\right)^2\right)
\end{aligned}$$

with $N_0^h = 0$, $\lim_{t\to\infty} \lambda_t^h N_t^h = 0$.

Proof. (**Proposition 1**) ■

We define the HJB equation,

$$\begin{aligned}
H_{mon}^h\left(N_t^h, p_t^h, A_t^h, t\right) &= G^h\left(p_t^h, A_t^h, N_t^h, t\right) + \lambda_t^h\, g^h\left(p_t^h, A_t^h, N_t^h, t\right), \qquad (2.13) \\
G^h\left(p_t^h, A_t^h, N_t^h, t\right) &= e^{-it}\left(\left(p_t^h - c_t^h\right)N_t^h - A_t^h\right), \\
g^h\left(p_t^h, A_t^h, N_t^h, t\right) &= \left[\left(a + u\frac{N_t^h}{M}\right)\left(M - N_t^h\right) - k_h N_t^h\right] \\
&\quad \times\left(1 + b\ln A_t^h - m\left(e^{rt}\bar{p} - p_t^h\right)^2\right)
\end{aligned}$$

The maximum principle conditions are:

$$\begin{aligned}
0 &= \frac{\partial}{\partial A^h}H_{mon}^h = \frac{\partial}{\partial A^h}G_t^h + \lambda_t^h\frac{\partial}{\partial A^h}g_t^h, \\
0 &= \frac{\partial}{\partial p^h}H_{mon}^h = \frac{\partial}{\partial p^h}G_t^h + \lambda_t^h\frac{\partial}{\partial p^h}g_t^h, \\
\dot{N}_t^h &= \frac{\partial}{\partial \lambda^h}H_{mon}^h = g^h\left(p_t^h, A_t^h, N_t^h, t\right), \\
\dot{\lambda}_t^h &= -\frac{\partial}{\partial N^h}H_{mon}^h = -\frac{\partial}{\partial N^h}G_t^h - \lambda_t^h\frac{\partial}{\partial N^h}g_t^h,
\end{aligned}$$

with $N_0^h = 0$, $\lambda_t^h \to 0$. From the first condition:

$$\begin{aligned}
0 &= -e^{-it} + \lambda_t^h\left[\left(a + u\frac{N_t^h}{M}\right)\left(M - N_t^h\right) - k_h N_t^h\right]\frac{b}{A_t^h} \\
&\Rightarrow\ A_t^h = be^{it}\lambda_t^h\left[\left(a + u\frac{N_t^h}{M}\right)\left(M - N_t^h\right) - k_h N_t^h\right]
\end{aligned}$$

The second condition leads to:

$$
\begin{aligned}
0 &= e^{-it} N_t^h + \lambda_t^h 2m \left(e^{rt}\bar{p} - p_t^h\right) \left[\left(a + u\frac{N_t^h}{M}\right)\left(M - N_t^h\right) - k_h N_t^h\right] \\
p_t^h &= e^{rt}\bar{p} + \frac{e^{-it} N_t^h}{\lambda_t^h 2m \left[\left(a + u\frac{N_t^h}{M}\right)\left(M - N_t^h\right) - k_h N_t^h\right]}
\end{aligned}
$$

From the third and fourth conditions:

$$
\begin{aligned}
\dot{N}_t^h &= \left[\left(a + u\frac{N_t^h}{M}\right)\left(M - N_t^h\right) - k_h N_t^h\right]\left(1 + b\ln A_t^h - m\left(e^{rt}\bar{p} - p_t^h\right)^2\right) \\
\dot{\lambda}_t^h &= -e^{-it}\left(p_t^h - c_t^h\right) \\
&\quad -\lambda_t^h \left(\frac{u}{M}\left(M - N_t^h\right) - \left(a + u\frac{N_t^h}{M}\right) - k_h\right)\left(1 + b\ln A_t^h - m\left(e^{it}\bar{p} - p_t^h\right)^2\right)
\end{aligned}
$$

$\square$

The solution satisfies a BVP, with initial conditions and terminal conditions $\lim_{t\to\infty} \lambda_t^h N_t^h = 0$ (which are known as transversality conditions). It can be solved numerically with a Galerkin-Collocation method setting $\lambda_T^h N_T^h = 0$ for a large T. For details see Esteban-Bravo and Vidal-Sanz, (2007).

Proposition 2 *The open-loop Nash equilibrium pricing and advertising decisions when the license (Strategy 2) is implemented, are given by:*

$$
\begin{aligned}
p_t^l = \; & e^{rt}\overline{p} + e^{-it} N_t^l \Big/ \Big(-\lambda_{1t}^l 2d\left(\left(a + u\tfrac{N_t^h}{M} + g\tfrac{N_t^l}{M}\right)\left(M - N_t^h - N_t^l\right) - k_h N_t^h\right) \\
& + \lambda_{3t}^l 2\delta\left(\left(\alpha + v\tfrac{N_t^h}{M} + \gamma\tfrac{N_t^l}{M} + \chi\tfrac{l_t}{L}\right)\left(M - N_t^h - N_t^l\right) - k_l N_t^l\right)\Big) \\
p_t^h = \; & e^{rt}\overline{p} + e^{-it} N_t^h \Big/ \Big(\lambda_{1t}^h 2m\left[\left(a + u\tfrac{N_t^h}{M} + g\tfrac{N_t^l}{M}\right)\left(M - N_t^h - N_t^l\right) - k_h N_t^h\right] \\
& - \lambda_{3t}^h 2\mu\left[\left(\alpha + v\tfrac{N_t^h}{M} + \gamma\tfrac{N_t^l}{M} + \chi\tfrac{l_t}{L}\right)\left(M - N_t^h - N_t^l\right) - k_l N_t^l\right]\Big)
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
A_t^l = \quad & e^{it}\left[\lambda_{1t}^l s\left(a+u\frac{N_t^h}{M}+g\frac{N_t^l}{M}\right)+\lambda_{3t}^l\sigma\left(\alpha+\upsilon\frac{N_t^h}{M}+\gamma\frac{N_t^l}{M}+\chi\frac{l_t}{L}\right)\right] \\
& \times\left(M-N_t^h-N_t^l\right)-k_h s e^{it}\lambda_{1t}^l N_t^h-k_l\sigma e^{it}\lambda_{3t}^l N_t^l \\
A_t^h = \quad & e^{it}\left[b\lambda_{1t}^h\left(a+u\frac{N_t^h}{M}+g\frac{N_t^l}{M}\right)+\lambda_{3t}^h\beta\left(\alpha+\upsilon\frac{N_t^h}{M}+\gamma\frac{N_t^l}{M}+\chi\frac{l_t}{L}\right)\right] \\
& \times\left(M-N_t^h-N_t^l\right)-k_h b e^{it}\lambda_{1t}^h N_t^h-k_l\beta e^{it}\lambda_{3t}^h N_t^l,
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
r_t^h = \quad & e^{rt}\bar{r}+\frac{e^{-it}N_t^l}{\lambda_{2t}^h 2z_6\left[\left(z_1+z_2\frac{l_t}{L}+z_3\frac{N_t^l}{M}\right)(L-l_t)-z_4 l_t\right]} \\
f_t^h = \quad & e^{rt}\bar{f}+\frac{e^{-it}l_t}{\lambda_{2t}^h 2z_5\left[\left(z_1+z_2\frac{l_t}{L}+z_3\frac{N_t^l}{M}\right)(L-l_t)-z_4 l_t\right]}
\end{aligned}
\tag{2.16}
$$

and the variables N_t^h, N_t^l, l_t, and $\lambda_{1t}^l, \lambda_{2t}^l, \lambda_{3t}^l, \lambda_{1t}^h, \lambda_{2t}^h, \lambda_{3t}^h$ are the solution to the BVP defined by equations (2.2), (2.3), (2.4), the co-state equations for the licensee

$$
\begin{aligned}
\dot{\lambda}_{1t}^l = \quad & -\lambda_{1t}^l\left[u\frac{1}{M}\left(M-N_t^h-N_t^l\right)-\left(a+u\frac{N_t^h}{M}+g\frac{N_t^l}{M}\right)-k_h\right]W_t^h\left(A_t^h,A_t^l,p_t^h,p_t^l\right) \\
& -\lambda_{3t}^l\left[\frac{\upsilon}{M}\left(M-N_t^h-N_t^l\right)-\left(\alpha+\upsilon\frac{N_t^h}{M}+\gamma\frac{N_t^l}{M}+\chi\frac{l_t}{L}\right)\right]W_t^l\left(A_t^h,A_t^l,p_t^h,p_t^l\right), \\
\dot{\lambda}_{2t}^l = \quad & e^{-it}f_t-\lambda_{2t}^l\left[\frac{z_2}{L}\left(L-l_t\right)-\left(z_1+z_2\frac{l_t}{L}+z_3\frac{N_t^l}{M}\right)-z_4\right]L_t\left(f_t^h,r_t^h\right) \\
& -\lambda_{3t}^l\frac{\chi}{L}\left(M-N_t^h-N_t^l\right)W_t^l\left(A_t^h,A_t^l,p_t^h,p_t^l\right), \\
\dot{\lambda}_{3t}^l = \quad & -e^{-it}\left(p_t^l-c_t^l-r_t^h\right) \\
& -\lambda_{1t}^l\left(g\frac{1}{M}\left(M-N_t^h-N_t^l\right)-\left(a+u\frac{N_t^h}{M}+g\frac{N_t^l}{M}\right)\right)W_t^h\left(A_t^h,A_t^l,p_t^h,p_t^l\right) \\
& -\lambda_{2t}^l\frac{z_3}{M}\left(L-l_t\right)\ L_t\left(f_t^h,r_t^h\right) \\
& -\lambda_{3t}^l\left(\frac{\gamma}{M}\left(M-N_t^h-N_t^l\right)-\left(\alpha+\upsilon\frac{N_t^h}{M}+\gamma\frac{N_t^l}{M}+\chi\frac{l_t}{L}\right)-k_l\right)W_t^l\left(A_t^h,A_t^l,p_t^h,p_t^l\right),
\end{aligned}
$$

and the co-state equations for the IPR holder

$$\begin{aligned}
\dot{\lambda}_{1t}^{h} = \quad & -e^{-it}\left(p_t^h - c_t^h\right) \\
& -\lambda_{1t}^{h}\left(\left[\tfrac{u}{M}\left(M - N_t^h - N_t^l\right) - \left(a + u\tfrac{N_t^h}{M} + g\tfrac{N_t^l}{M}\right) - k_h\right] W_t^h\left(A_t^h, A_t^l, p_t^h, p_t^l\right)\right) \\
& -\lambda_{3t}^{h}\left[\tfrac{\upsilon}{M}\left(M - N_t^h - N_t^l\right) - \left(\alpha + \upsilon\tfrac{N_t^h}{M} + \gamma\tfrac{N_t^l}{M} + \chi\tfrac{l_t}{L}\right)\right] W_t^l\left(A_t^h, A_t^l, p_t^h, p_t^l\right), \\
\dot{\lambda}_{2t}^{h} = \quad & -e^{-it} f_t^h \\
& -\lambda_{2t}^{h}\left(\tfrac{z_2}{L}\left(L - l_t\right) - \left(z_1 + z_2\tfrac{l_t}{L} + z_3\tfrac{N_t^l}{M}\right) - z_4\right) L_t\left(f_t^h, r_t^h\right) \\
& -\lambda_{3t}^{h}\tfrac{\chi}{L}\left(M - N_t^h - N_t^l\right) W_t^l\left(A_t^h, A_t^l, p_t^h, p_t^l\right), \\
\dot{\lambda}_{3t}^{h} = \quad & -e^{-it} r_t^h \\
& -\lambda_{1t}^{h}\left(\left(\tfrac{g}{M}\left(M - N_t^h - N_t^l\right) - \left(a + u\tfrac{N_t^h}{M} + g\tfrac{N_t^l}{M}\right)\right) W_t^h\left(A_t^h, A_t^l, p_t^h, p_t^l\right)\right) \\
& -\lambda_{2t}^{h}\tfrac{z_3}{M}\left(L - l_t\right)\; L_t\left(f_t^h, r_t^h\right) \\
& -\lambda_{3t}^{h}\left[\tfrac{\gamma}{M}\left(M - N_t^h - N_t^l\right) - \left(\alpha + \upsilon\tfrac{N_t^h}{M} + \gamma\tfrac{N_t^l}{M} + \chi\tfrac{l_t}{L}\right) - k_l\right] W_t^l\left(A_t^h, A_t^l, p_t^h, p_t^l\right),
\end{aligned}$$

with initial values $N_0^h = N_0^l = l_0 = 0$, *and terminal conditions* $\lim_{t\to\infty}\lambda_{1t}^l N_t^h = 0$, $\lim_{t\to\infty}\lambda_{2t}^l l_t = 0$,$\lim_{t\to\infty}\lambda_{3t}^l N_t^l = 0$, $\lim_{t\to\infty}\lambda_{1t}^h N_t^h = 0$, $\lim_{t\to\infty}\lambda_{2t}^h l_t = 0$,$\lim_{t\to\infty}\lambda_{3t}^h N_t^l = 0$.

The proof of Proposition 2 is similar to that of Proposition 1. Next, we provide the first order conditions for the optimal policies under piracy, based on Strategy 3.

Proposition 3 *If an IPR holder faces piracy competition (Strategy 3), the open-loop*

Nash equilibrium pricing and advertising decisions are given by:

$$
\begin{aligned}
p_t^c = \quad & e^{rt}\bar{p}^c + e^{-it}N_t^c \Big/ \Big(-\lambda_{1t}^c 2d \Big[\Big(a + u\tfrac{N_t^h}{M^h} + g\tfrac{N_t^c}{M^c}\Big)\left(M^h - N_t^h\right) - k_h N_t^h\Big] \\
& + \lambda_{3t}^c 2\delta \Big[\Big(\alpha + \upsilon\tfrac{N_t^h}{M^h} + \gamma\tfrac{N_t^c}{M^c} + \chi\tfrac{l_t^c}{L}\Big)\left(M^c - N_t^c\right) - k_c N_t^c\Big]\Big), \\
p_t^h = \quad & e^{rt}\bar{p} + e^{-it}N_t^h \Big/ \Big(\lambda_1^h 2m \Big[\Big(a + u\tfrac{N_t^h}{M^h} + g\tfrac{N_t^c}{M^c}\Big)\left(M^h - N_t^h\right) - k_h N_t^h\Big] \\
& - \lambda_{3t}^h 2\mu \Big[\Big(\alpha + \upsilon\tfrac{N_t^h}{M^h} + \gamma\tfrac{N_t^c}{M^c} + \chi\tfrac{l_t^c}{L}\Big)\left(M^c - N_t^c\right) - k_c N_t^c\Big]\Big), \\
A_t^c = \quad & s\lambda_{1t}^c e^{it}\Big[\Big(a + u\tfrac{N_t^h}{M^h} + g\tfrac{N_t^c}{M^c}\Big)\left(M^h - N_t^h\right) - k_h N_t^h\Big] \\
& + \sigma\lambda_{3t}^c e^{it}\Big[\Big(\alpha + \upsilon\tfrac{N_t^h}{M^h} + \gamma\tfrac{N_t^c}{M^c} + \chi\tfrac{l_t^c}{L}\Big)\left(M^c - N_t^c\right) - k_c N_t^c\Big], \\
A_t^h = \quad & b\lambda_{1t}^h e^{it}\Big[\Big(a + u\tfrac{N_t^h}{M^h} + g\tfrac{N_t^c}{M^c}\Big)\left(M^h - N_t^h\right) - k_h N_t^h\Big] \\
& + \beta\lambda_{3t}^h e^{it}\Big[\Big(\alpha + \upsilon\tfrac{N_t^h}{M^h} + \gamma\tfrac{N_t^c}{M^c} + \chi\tfrac{l_t^c}{L}\Big)\left(M^c - N_t^c\right) - k_c N_t^c\Big].
\end{aligned}
$$

where $N_t^h, N_t^c, l_t^c, \lambda_{1t}^h, \lambda_{2t}^h, \lambda_{3t}^h, \lambda_{1t}^c, \lambda_{2t}^c, \lambda_{3t}^c$ are the solution to the BVP defined by (2.5), (2.6), and the respective co-state equations

$$
\begin{aligned}
\dot{\lambda}_{1t}^c = \quad & -\lambda_{1t}^c\Big[u\tfrac{1}{M^h}\left(M^h - N_t^h\right) - \Big(a + u\tfrac{N_t^h}{M^h} + g\tfrac{N_t^c}{M^c}\Big) - k_h\Big]\; W_t^h\left(A_t^h, A_t^c, p_t^h, p_t^c\right) \\
& - \lambda_{3t}^c \upsilon \tfrac{1}{M^h}\left(M^c - N_t^c\right)\; W_t^c\left(A_t^h, A_t^c, p_t^h, p_t^c\right), \\
\dot{\lambda}_{2t}^c = \quad & -\lambda_{2t}^c\Big[z_2\tfrac{1}{L}\left(L - l_t^c\right) - \Big(z_1 + z_2\tfrac{l_t^c}{L} + z_3\tfrac{N_t^c}{M^c}\Big)\left(L - l_t^c\right) - z_4\Big] \\
& - \lambda_{3t}^c \chi \tfrac{1}{L}\left(M^c - N_t^c\right) W_t^c\left(A_t^h, A_t^c, p_t^h, p_t^c\right), \\
\dot{\lambda}_{3t}^c = \quad & -e^{-it}\left(p_t^c - c_t^c\right) - \lambda_{1t}^c g\tfrac{1}{M^c}\left(M^h - N_t^h\right) W_t^h\left(A_t^h, A_t^c, p_t^h, p_t^c\right) - \lambda_{2t}^c z_3\tfrac{1}{M^c}\left(L - l_t^c\right) \\
& - \lambda_{3t}^c\Big[\gamma\tfrac{1}{M^c}\left(M^c - N_t^c\right) - \Big(\alpha + \upsilon\tfrac{N_t^h}{M^h} + \gamma\tfrac{N_t^c}{M^c} + \chi\tfrac{l_t^c}{L}\Big) - k_c\Big]\; W_t^c\left(A_t^h, A_t^c, p_t^h, p_t^c\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\lambda}_{1t}^{h} = \quad & -e^{-it}\left(p_t^h - c_t^h\right) - \lambda_{1t}^h \left[u\tfrac{1}{M^h}\left(M^h - N_t^h\right) - \left(a + u\tfrac{N_t^h}{M^h} + g\tfrac{N_t^c}{M^c}\right) - k_h\right] \\
& \cdot W_t^h\left(A_t^h, A_t^c, p_t^h, p_t^c\right) \\
& -\lambda_{3t}^h \upsilon \tfrac{1}{M^h}\left(M^c - N_t^c\right)\ W_t^c\left(A_t^h, A_t^c, p_t^h, p_t^c\right), \\
\dot{\lambda}_{2t}^{h} = \quad & -\lambda_{2t}^h\left[z_2\tfrac{1}{L}\left(L - l_t^c\right) - \left(z_1 + z_2\tfrac{l_t^c}{L} + z_3\tfrac{N_t^c}{M^c}\right)\left(L - l_t^c\right) - z_4\right] \\
& -\lambda_{3t}^h \chi \tfrac{1}{L}\left(M^c - N_t^c\right) W_t^c\left(A_t^h, A_t^c, p_t^h, p_t^c\right), \\
\dot{\lambda}_{3t}^{h} = \quad & -\lambda_{1t}^h g \tfrac{1}{M^c}\left(M^h - N_t^h\right) W_t^h\left(A_t^h, A_t^c, p_t^h, p_t^c\right) - \lambda_{2t}^h z_3 \tfrac{1}{M^c}\left(L - l_t^c\right) \\
& -\lambda_{3t}^h\left[\gamma\tfrac{1}{M^c}\left(M^c - N_t^c\right) - \left(\alpha + \upsilon\tfrac{N_t^h}{M^h} + \gamma\tfrac{N_t^c}{M^c} + \chi\tfrac{l_t^c}{L}\right) - k_c\right]\ W_t^c\left(A_t^h, A_t^c, p_t^h, p_t^c\right)
\end{aligned}
$$

with $N_0^h = 0$, $N_0^c = 0, l_0^c = 0, \lim_{t\to\infty}\lambda_{1t}^h N_t^h = 0$, $\lim_{t\to\infty}\lambda_{2t}^h l_t^c = 0$, $\lim_{t\to\infty}\lambda_{3t}^h N_t^c = 0$, $\lim_{t\to\infty}\lambda_{1t}^c N_t^h = 0$, $\lim_{t\to\infty}\lambda_{2t}^c l_t^c = 0$, $\lim_{t\to\infty}\lambda_{3t}^c N_t^c = 0$.

The proof of Proposition 3 is similar to that of Proposition 1.

Chapter 3

Riding Successive Product Diffusion Waves. Building a Tsunami via Upgrade–Rebate Programs

3.1 Introduction

In this paper we study how to manage sales of durable new products under successive generation diffusion waves with trade-in rebates. If the durability of the product is large relative to the introduction times of successive generations, they cannibalize each other as firms often sell inventory of old generation goods which were leftover from previous periods simultaneously with newer ones. The older generations are often perceived to be of lower quality, and firms usually sell it at a reduced price (Koenigsberg and Ferguson, 2007). Some potential customers from the old generation can upgrade to new one, but others could buy the old one at a lower price, and this can affect a large share of customers. The innovation diffusion literature has suggested that fear of obsolescence may cause some consumers to refuse to buy technological products from the first generations (Cohen et al. 1996; Dhebar 1996; Venkatesh and Brown 2001), waiting for new products and accumulate old potential buyers for new generations (Putsis 1993). Second hand markets for used durable goods reduce consumers' threat of leapfrogging, but on the other hand generate additional competition from old products that reenter the market (Desai, Koenigsberg, and Purohit, 2007). Some companies buy-back in second-hand markets and remanufacture the product gaining a competitive advantage (Heese et al. 2005). Upgrade–Rebates can also be used to disable second-hand market for the old versions, using buybacks to retire the old units from the market (see Levinthal and Purohit, 1989).

Managing diffusion waves for successive product generations implies that marketing managers should try to dissuade some customers to jump across successive generation waves optimally for the company profits. In addition, companies should consider some strategy to lessen the regret of consumers who have bought old-generation products and persuade them to upgrade to new generations in future (Dhebar 1996). In this paper we discuss how Upgrade–Rebates (also known as trade-in rebates) can be used to reintegrate owners of old versions of the product to the market and therefore increase current sales. An upgrade-rebate program allows customers who own an old generation version of the product to trade it in for a product belonging to a later generation at a discounted price. This is also an instrument to make a credible threat that if the consumers do not buy the first generation product, they will face a relatively higher price for the new product in subsequent generations.

Upgrade-rebates are implemented by companies in different industries such as software, electrical appliances, cars, etc. For instance, currently Hewlett-Packard allows its customers to trade-in different types of products and brands for its new HP products[1]. Xerox provides up to $600 rebate to trade in old product of own and competitive brands for its new product models[2]. Similar strategies are also implemented by Motorola and Sony, among others. Governments often adopt trade-in strategies. Some examples are the Car Allowance Rebate System (CARS) program launched out by the US Congress

[1] http://www.hp.com/united-states/tradein/home_flash.html
[2] http://xerox.tradeups.com/Customers/17/Default.aspx

to increase automotive sales and aid the environment and the Home Renovation Rebate in Canada in the form of a tax credit.

Trade-ins can be considered as a strategic link between reverse and direct channels management, where the customer bears part of the reverse-logistics and the returned product is not remarketed (see Ray, Boyaci and Aras, 2005). Upgrade-rebates have been considered from a pricing perspective. Notice that new customers pay more than those who upgrade an old product, and these rebates can be thought of as a price discrimination based on consumers' purchase history. This has been studied by Lee and Lee (1998), Levinthal and Purohit (1989) in second hand markets and Fudenberg and Tirole (1998) in a more general setup. Ackere and Reyniers (1995) consider quasidurable goods. But in these models the results depend heavily on parametric assumptions, generally in a two-period framework. The rich dynamic perspective, where upgrade-rebates are used to control the whole diffusion of successive generations, is hitherto unexplored. In this paper we study this problem in a context without a second-hand market, and compute the optimal rebates policy to ride the successive diffusion waves.

The organization of the remainder of the paper is the following. In the next section we present a general model of successive product generations diffusion, the inclusion of the upgrade-rebate program into the benchmark model, and the optimal solution of the problem. In Section 3 we provide some simulations with numerical solutions and results. In Section 4 we provide an empirical application of the upgrade-rebate model to the automobile industry in Spain. Finally, we conclude the paper with general discussion, pointing out some limitations and suggestions for future research.

3.2 A Successive Generations Diffusion Model

In this section we present a diffusion model for successive product generations. We build up on previous work on successive generation models, such as Norton and Bass (1987, 1992), Mahajan and Muller (1996), and Stremersch, Muller and Peres (2010). We first introduce the benchmark model without upgrade-rebates, and then we discuss how the model can accommodate this type of promotion.

3.2.1 The Benchmark Model

Let us consider K product generations, the $i-th$ one is introduced at time τ_i with a market potential m_i. After the entry of generation i, the older generations j, where $j<i$, can still continue to acquire some customers from $m_j, m_{j-1}, m_{j-2}, ..., m_1$, but more importantly, the new generation attracts natural upgraders from the older generations $m_{i-1}, m_{i-2}, ...m_1$. The expression of natural upgraders is used to point out that they are not attracted by a promotion, but are those which would have bought the old generation had the new generation(s) not been introduced. The model allows leapfrogging (i.e. upgrading beyond the subsequent generation) and the specification can be easily adapted to durable and non durable products. In particular, for non-durable products we consider sales defined as

$$\begin{aligned} \begin{pmatrix} S_1(t) \\ S_2(t) \\ \vdots \\ S_K(t) \end{pmatrix} &= \begin{pmatrix} f_{11}(t) & 0 & \dots & 0 \\ f_{12}(t) & f_{22}(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{1K}(t) & f_{2K}(t) & \dots & f_{KK}(t) \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_K \end{pmatrix} \Leftrightarrow \\ \mathbf{S}(t) &= \mathbf{f}(t)' \ m \end{aligned} \tag{3.1}$$

where $m=(m_1, ..., m_K)'$, and $\mathbf{f}(t)$ is a diffusion matrix with generic element $f_{ij}(t) \in (0,1)$ describing the proportion of potential customers from m_i purchasing product from generation j at time t. Obviously $f_{ii}(t)$ is the net penetration of generation m_i at time t removing the upgraders. The transposed diffusion matrix $\mathbf{f}(t)'$ is lower triangular to ensure that old generation products do not attract customers from the new generations, and the elements of each column of $\mathbf{f}(t)'$ sum less than one (i.e., $\sum_{j=i}^{K} f_{ij}(t) \leq 1$ for all $i=1...K$), to ensure that the summation of sales drawn from potential customers of generation i do not exceed the potential market ceiling m_i. It implies that $\sum_{j=i}^{K} \dot{f}_{ij}(t) \leq 0$

for all $i = 1...K$.

This is a flexible structure, and several specifications can be considered for $\dot{f}_{ij}$. In particular, we consider that the departing rate from potential customers of generation i to purchase product generation j is given by

$$\dot{f}_{ij} = \begin{cases} \left(p_{ij} + q'_{ij} f_j(t)\right)\left(1 - \sum_{l=i}^{K} f_{il}(t)\right), \quad f_{ij}(\tau_j) = 0, & j \geq i \\ 0 & j < i \end{cases} \tag{3.2}$$

where $f_j(t) = (f_{1j}(t), \ldots, f_{jj}(t), 0, \ldots, 0)'$ is the column j of matrix $\mathbf{f}(t)$ and q_{ij} is a vector with all the coefficients equal to zero after row j, meaning that the upgrading from i to j is accelerated when f_{jj} is high and when more people upgrade from old generations to j. The $\left(1 - \sum_{l=i}^{K} f_{il}(t)\right)$ factor ensures that the columns of $\mathbf{f}(t)'$ total at most 1 at any time. Notice that we can consider more complex structures for defections in this market, e.g. considering a generation 0, with $S_0(t)$ the number of defections and a row $f_{j0}(t)$ in $\mathbf{f}(t)'$ accounting for the attractions from all generations to this event.

The previous model is appropriated for non-durable products so that the adopters $\mathbf{f}(t)' m$ at time t are the current sales (except for the defections, if they had been considered). If instead of non-durables we model the diffusion of durable products, we can simply replace (3.1) by the expression

$$\mathbf{S}(t) = \dot{\mathbf{f}}(t)' m. \tag{3.3}$$

meaning that sales at time t are given by the change in penetration rates weighted by the market potentials. Notice that the vector $\mathbf{f}(t)' 1$ is the penetration percentage of each generation at time t. In both cases, durables and non-durables, we can introduce controls on the diffusion matrix $\mathbf{f}(t)$ analogously to the Horsky and Simon (1983) model, or multiplicatively when these controls also affect the market potentials m. Note also that semi-durables could be handled in the benchmark setting

$$\mathbf{S}(t) = \dot{\mathbf{f}}(t)' m + \int_0^t \dot{\mathbf{f}}(s)' m \; \Delta(t-s)\, ds$$

where $\Delta(\tau) = dig\{\Delta_i(\tau)\}$ is a diagonal matrix with the probabilities of a product of generation i perishing after τ periods.

3.2.2 The Model With an Upgrade–Rebate Program

Let us define $r(t)$ as the cumulative Upgrade–Rebate matrix, i.e. the element $r_{ij}(t) \in (0,1)$ is the *fraction* of individuals owning a product from generation i who have upgraded it to a product of generation j by time t. The fraction of individuals upgrading at time t is $\dot{\mathbf{r}}(t)$. Theis matrix is lower triangular, as not all types of upgrades are pursued by the company, for example

$$\mathbf{r}(t)' = \begin{pmatrix} 0 & 0 & \dots & 0 \\ r_{12}(t) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_{1K}(t) & r_{2K}(t) & \dots & 0 \end{pmatrix}.$$

if the company just allows changes from old generation purchases to new generations and not vice versa. In practice, this matrix is often spare since just a few elements are non-zero as only upgrades between consecutive generations are permitted, but allowing most of the old trade-up pair combinations the optimization analysis is more flexible. Similar to the benchmark model, the elements of each column of $\mathbf{r}(t)$ sum at most one (i.e., $\sum_{j=i+1}^{K} r_{ij}(t) \leq 1$ for all $i = 1...K$), to ensure that the summation of upgraders from generation i to other generations does not exceed one, this implies that $\sum_{j=i+1}^{K} \dot{r}_{ij}(t) \leq 0$ for $i = 1...K$. Once again this is a flexible structure, and several specifications can be considered for $\dot{r_{ij}}$. In particular, we can consider that for $j > i$

$$\begin{aligned}
\dot{r_{ij}}(t) &= \left(\gamma_{ij} + \beta_{ij}' r_j(t)\right)\left(1 - \sum_{l=i+1}^{K} r_{il}(t)\right)\phi_{ij} D_{ij}(t) \\
r_{ij}(\tau_j) &= 0.
\end{aligned}$$

and $\dot{r_{ij}}(t) = 0$ for $i \geq j$, where $r_j(t) = \left(r_{1j}(t), \ldots, r_{(j-1)j}(t), 0, \ldots, 0\right)'$ is the column j of matrix $\mathbf{r}(t)$, and $D_{ij}(t) \geq 0$ are rebates obtained on the purchase of j when the product i is returned.

The diffusion $\mathbf{f}(t)$ of the different generations will be affected by the upgrades $\mathbf{r}(t)$. The upgrade–rebate program also provides a response to the consumers threat of leapfrogging, and the customers who use a rebate spread word-of-mouth in a different way than those who do not upgrade through this promotion. In particular we replace (3.2) by,

$$\dot{f}_{ij} = \left(p_{ij} + q_{ij}' f_j(t) + r_j(t)' f(t) b_{ij}\right)\left(1 - \sum_{l=i}^{K} f_{il}(t)\right)\left(1 - \theta_{ij} P_j(t)\right),$$

for $j \geq i$ and $\dot{f}_{ij} = 0$ when $j < i$, with $f_{ij}(\tau_j) = 0$, where q_{ij} is a column vector with all the coefficients equal to zero after row j, $P_j(t)$ are prices of generation j, and b_{ij} is a column vector with zero elements after j. For simplicity, we assume multiplicative linear prices (e.g. Dockner and Jorgensen 1988).

Next we discuss the sales behavior. Recall that $\mathbf{f}(t)' m$ is the vector of cumulated sales from the different generations up to time t. The cumulative sales caused by the Upgrade–Rebate Program at time t are (as a column including all generation):

$$\begin{aligned}
\mathbf{r}(t)' \mathbf{f}(t)' m &= \int_0^t \left(\dot{\mathbf{r}}(s)' \mathbf{f}(s)' m + \mathbf{r}(s)' \dot{\mathbf{f}}(s)' m\right) ds \\
&= \int_0^t \left(\mathbf{f}(s)\dot{\mathbf{r}}(s) + \dot{\mathbf{f}}(s)\mathbf{r}(s)\right)' m ds,
\end{aligned}$$

(the proof is immediate as $\int_0^t \left(\mathbf{f}(s)\dot{\mathbf{r}}(s)\right)' m ds = (\mathbf{f}(t)\mathbf{r}(t))' m - \int_0^t \left(\dot{\mathbf{f}}(s)\mathbf{r}(s)\right)' m ds$ integrating by parts). Therefore, the instantaneous Upgrade–Rebate sales at time t is the derivative of $\mathbf{r}(t)' \mathbf{f}(t)' m$ which is equal to $\left(\mathbf{f}(t)\dot{\mathbf{r}}(t) + \dot{\mathbf{f}}(t)\mathbf{r}(t)\right)' m$; i.e. those types of current sales are decomposed as the sum of sales to old buyers who upgrade now $\dot{\mathbf{r}}(t)' \mathbf{f}(t)' m$, and sales of current buyers of old versions who immediately choose to upgrade $r(t)' \dot{\mathbf{f}}(t)' m$. In order to ban immediate upgrades some companies could consider a constrained rebate structured with $r(t)$ orthogonal to $\dot{\mathbf{f}}(t)$, but this could be suboptimal and we will not impose it in our basic structure.

Finally, the total sales at time t in all product generations are the sum of regular sales of the durable products as in (3.3), and the second summand has sales drawn from the Upgrade–Rebate promotion

$$S(t) = \dot{\mathbf{f}}(t)' m + \left(\mathbf{f}(t)\dot{r}(t) + \dot{\mathbf{f}}(t) r(t)\right)' m$$

The company profits at time t are

$$\begin{aligned} \Pi(t) &= m' \left(\dot{\mathbf{f}}(t) + \mathbf{f}(t)\dot{r}(t) + \dot{\mathbf{f}}(t)\mathbf{r}(t)\right)(P(t) - c(t)) \\ &\quad - m' \left(\mathbf{f}(t)\mathbf{D}(t)\dot{r}(t) + \dot{\mathbf{f}}(t)\mathbf{D}(t)\mathbf{r}(t)\right)\mathbf{1}, \end{aligned}$$

where $\mathbf{D}(t)$ is the rebates matrix and $(P(t) - c(t))$ is a vector with the unit margins for each generation and $\mathbf{1}$ is a vector of ones.

3.2.3 The Optimal Behavior

Assuming a durable-goods monopolist who controls the diffusion of all generations, the company profits are obtained by maximizing:

$$\begin{aligned} \Pi &= \int_0^\infty e^{-rt}\Pi(t)\, dt \\ &= \int_0^\infty e^{-rt}\left(m'\left(\dot{\mathbf{f}}(t) + \mathbf{f}(t)\dot{r}(t) + \dot{\mathbf{f}}(t)\mathbf{r}(t)\right)(P(t) - c(t))\right. \\ &\quad \left. - m'\left(\mathbf{f}(t)\mathbf{D}(t)\dot{\mathbf{r}}(t) + \dot{\mathbf{f}}(t)\mathbf{D}(t)\mathbf{r}(t)\right)\mathbf{1}\right) dt, \end{aligned}$$

subject to the state equations and the initial values $f_{ij}(\tau_j) = r_{ij}(\tau_j) = 0$ for each $i \leq j$.

In order to write the first-order conditions of the optimal control problem, it is convenient to write all the state variables as vectors instead of matrices. We define $\dot{f}$, $\dot{r}_t$, f_t, r_t, D_t the half-vectorization[3] of matrices $\dot{\mathbf{f}}(t)$, $\dot{\mathbf{r}}(t)$, $\mathbf{f}(t)$, $\mathbf{r}(t)$, $\mathbf{D}(t)$, respectively; introducing a compact notation for profits and state equations,

$$\begin{aligned} \Pi_t &= \Pi(f_t, r_t, P_t, D_t) \\ \dot{f}_t &= \psi(f_t, r_t, P_t), \\ \dot{r}_t &= \rho(f_t, r_t, D_t). \end{aligned}$$

Let the Hamiltonian function H of the optimal control be given by:

$$H_t(f_t, r_t, P_t, D_t, \lambda_t, \mu_t) = \Pi(f_t, r_t, P_t, D_t) + \lambda_t \cdot \psi(f_t, r_t, P_t) + \mu_t \cdot \rho(f_t, r_t, D_t),$$

where λ_t and μ_t be the multiplier vectors associated with the state variables f_t and r_t, respectively. The maximum principle conditions are (in compact notation)

$$\begin{array}{lll} \frac{\partial H_t}{\partial D_t} = 0, & \frac{\partial H_t}{\partial P_t} = 0 & \text{for any } i < j, \\ \dot{\lambda}_t = -\frac{\partial H_t}{\partial f_t}, & & \text{for any } i \leq j, \\ \dot{\mu}_t = -\frac{\partial H_t}{\partial r_t} & & \text{for any } i < j, \end{array}$$

together with initial conditions at introduction times τ_i, and transversality conditions of λ_t, μ_t tending to zero (numerically we equalize them to zero for large t). Also, we require that the marginal valuation of the state variable(s) be the same evaluated before and after each τ_j (Amit, 1986; Kamien and Schwartz, 1991). This is a boundary value differential equation system, that must be particularized for each specific problem. In the

[3]The vectorization of an $n \times m$ matrix A, denoted by $vec(A)$, is the $nm \times 1$ column vector obtain by stacking the columns of the matrix A on top of one another proceeding from column 1 to column m. The half-vectorization $vech(A)$ of a $n \times n$ matrix A is the $n(n+1)/2 \times 1$ column vector obtained by vectorizing only the lower triangular part of A, i.e. staking the low-triangle part of the columns on top of one another from column 1 to m.

next sections we present some general examples and compute the numerical solutions.

3.3 Numerical Results

In order to analyze the effect of upgrade-rebate strategy on a firm's profits and sales, we study a monopolistic firm which manages successive product diffusion waves. As analytical solutions of the problems are not feasible, we solve the optimal problems numerically. We introduce several cases with increasing difficulty. For all the examples considered, we compare the numerical results of upgrade-rebate implementation with the benchmark case when the firm does not carry out the strategy.

3.3.1 Managing Two Generations

First we study the optimal behavior of the firm which manages the diffusion of two product generations assuming that the monopolistic firm takes the prices of the products as given. Next, relaxing this assumption, we examine a firm which optimally controls for prices and analyze the effect of upgrade-rebate strategy on firm profits and product diffusion waves. Finally, we discuss the scenario, when the firm provides the second generation product absolutely for free to the upgraders.

Two generations example with exogenous prices

We first present a simple case of a monopolistic firm which manages two waves of successive product generations. We assume that prices of the product generations are given exogenously. Before the second generation is introduced, i.e. $\tau_1 < t < \tau_2$, the state equation system is the following:

$$\begin{cases} \dot{f}_{11} = \left(p_{11} + q_{11} f_{11}(t)\right)\left(1 - f_{11}(t)\right)\left(1 - \theta_{11} P_1(t)\right), \\ \dot{f}_{12} = 0, \dot{f}_{22} = 0, \dot{r}_{12} = 0. \end{cases} \tag{3.4}$$

After the introduction of the second generation, $\tau_2 < t < T$, the system of state equations turns to:

$$\begin{cases} \dot{f}_{11} = \left(p_{11} + q_{11} f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right)\left(1 - \theta_{11} P_1(t)\right), \\ \dot{f}_{12} = \left(p_{12} + q_{12}^1 f_{12}(t) + q_{22}^1 f_{22}(t) + b_{12}^1 r_{12}(t) f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right)\left(1 - \theta_{12} P_2(t)\right) \\ \dot{f}_{22} = \left(p_{22} + q_{12}^2 f_{12}(t) + q_{22}^2 f_{22}(t) + b_{12}^2 r_{12}(t) f_{11}(t)\right)\left(1 - f_{22}(t)\right)\left(1 - \theta_{22} P_2(t)\right) \\ \dot{r}_{12} = \left(\gamma_{12} + \beta_{12} r_{12}(t)\right)\left(1 - r_{12}(t)\right)\phi_{12} D_{12}(t). \end{cases} \tag{3.5}$$

The company profits are given by

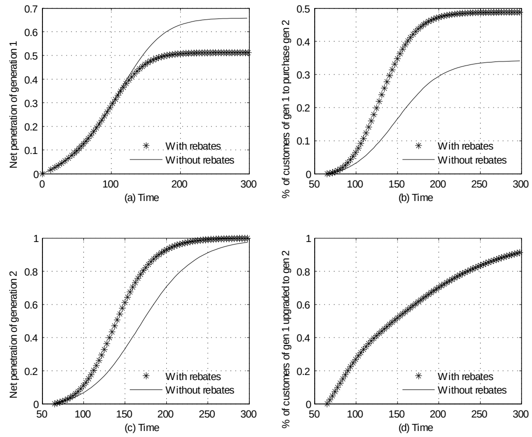
$$\begin{aligned} \Pi &= \int_0^\infty e^{-rt}\Pi(t)\,dt = \int_{\tau_1}^{\tau_2} e^{-rt}\Pi_1(t)\,dt + \int_{\tau_2}^\infty e^{-rt}\Pi_2(t)\,dt, \\ \Pi_1(t) &= \dot{f}_{11} m_1 \left(P_1(t) - c_1(t)\right) \\ \Pi_2(t) &= \left(\Pi_1(t) + \left(\dot{f}_{12} m_1 + \dot{f}_{22} m_2 + f_{11} m_1 \dot{r}_{12} + \dot{f}_{11} m_1 r_{12}\right)\left(P_2(t) - c_2(t)\right)\right. \\ &\quad \left. - \left(f_{11} m_1 \dot{r}_{12} + \dot{f}_{11} m_1 r_{12}\right) D_{12}(t)\right). \end{aligned}$$

The Hamiltonian conditions are given in Appendix A1. The system of resulting differential equations cannot be solved analytically. We compute the numerical solution for the following values of the parameters. We assume arbitrary total market sizes of $m_1 = 2000$ and $m_2 = 3000$ for the first and the second generation products, respectively. We take the following values for the coefficients of innovation: $p_{11} = 0.003$, $p_{12} = 0.001$, and $p_{22} = 0.003$; and coefficients of imitation: $q_{11} = 0.05$, $q_{12}^1 = q_{12}^2 = 0.05$, and $q_{22}^1 = q_{22}^2 = 0.05$. To capture the effect of price, we initially set $\theta_{11} = 0.002$, $\theta_{12} = 0.002$, $\theta_{22} = 0.002$. We assume that $b_{12}^1 = b_{12}^2 = 0.07$, which is the parameter showing how the product diffusion is affected by the proportion of upgraders through the rebate program. For the upgraders diffusion equation we set the parameters of innovation and imitation as $\gamma_{12} = 0.03$ and $\beta_{12} = 0.3$, respectively; and for the parameter of the sensitivity to the rebate discounts as $\phi_{12} = 0.0005$. We also set $P_1 = 300$, $P_2 = 300$, and $c_1 = 100$, $c_2 = 100$, i.e. the unit price and cost parameters for the first and second generation products. The discount rate is set to $r = 0.05$. We assume that the second generation is introduced at $\tau_2 = 65$ and compute the solution for a large time length ($T = 300$). The results are relatively robust when we tried other parametric values in the same ranges.

We insert the optimality conditions of the rebate strategy into eight differential equations (four equations of state variables given by equations (3.4) and (3.5), and four equations of multipliers defined by (3.6) and (3.7)-(3.10) defined before and after the introduction time of the second generation τ_2, respectively). We end up with a system of 8 differential equations with 8 unknowns, with 4 initial conditions, 4 boundary conditions, as well as continuity conditions of the state variables and the multipliers at τ_2. We solve the resulting BVP numerically, using the Galerkin-Collocation method (for an introduction see, for instance, Judd 1998).

Figure 1 shows the effect of the upgrade-rebate program on the sales diffusion for this set of parameters. From Figure 1 panel (c), the net penetration of the second generation product is significantly accelerated. The upgrade-rebate program also speeds up the natural upgrading process from generation 1 to generation 2 (Figure 1 panel (b)). The implementation of the program would result into a faster and eventually higher proportion of adopters buying the second generation product, but who would have bought the older generation had the second generation product not been introduced. On the other hand, these positive effects of the program on the sales diffusion of the second generation product are partially offset by a lower proportion of adopters of the first generation product (see Figure 1 panel (a)).

Figure 1. Effect of rebate program on diffusion



For this set of parameters, the optimal total discounted profit is $\Pi = 15369$ for the benchmark case and $\Pi = 16205$ if the company implements the trade-in strategy. Thus, the implementation of trade-in strategy increases the total discounted profits of the company by nearly 5.44%.

Figure 2 shows the optimal rebate amounts over time. As it can be seen, the optimal rebate strategy is to start at a somewhat high level and to decrease it gradually over time. The comparison of the optimal rebate strategy with the price value of the second generation reveals that initially for the company it is optimal to give the buyers of the first generation monetary incentives to buy the second generation product (i.e. $D_{12}(t) > P_2$

up to some t).

Figure 2. Optimal rebate amount

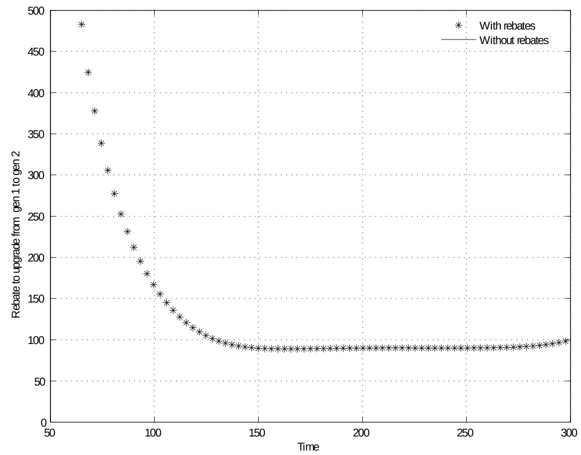
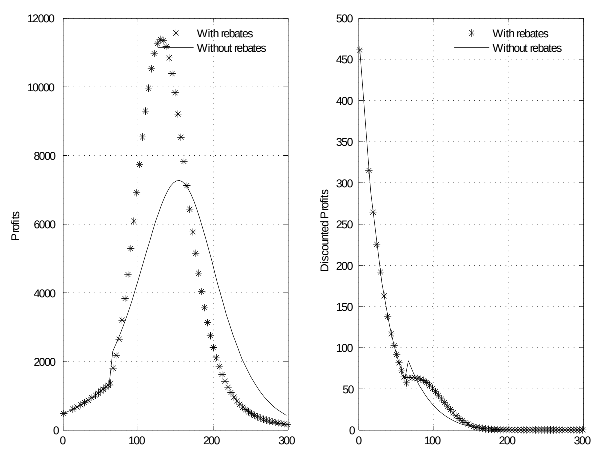


Figure 3 shows the optimal current profits (panel a) and discounted profits (panel b). Under both strategy scenarios (i.e. implementing or not upgrade-rebate policy) optimal current profits increase over time but after some time decline becoming closer to zero. However, when upgrade-rebate strategy is carried out, optimal profits peak earlier and at a higher level. Not surprisingly, discounted profits asymptotically tend to zero.

Figure 3. Optimal current and dicsounted profits



Two generation example with prices as controls

Next, we relax the assumption of exogenous prices and discuss a more realistic case of two product generations, when the firm maximizes its life-time profits optimally choosing prices and rebate amounts. We discuss the optimal conditions of the firm's maximization problem in the Appendix A2. We solved the system of differential equations using the Galerkin-Collocation method as in the previous case.

Figure 4. Effect of rebate program on diffusion

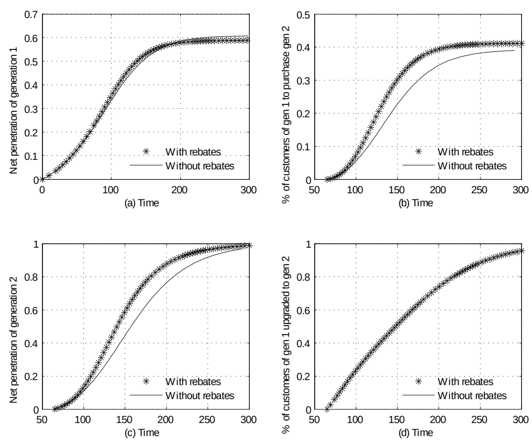
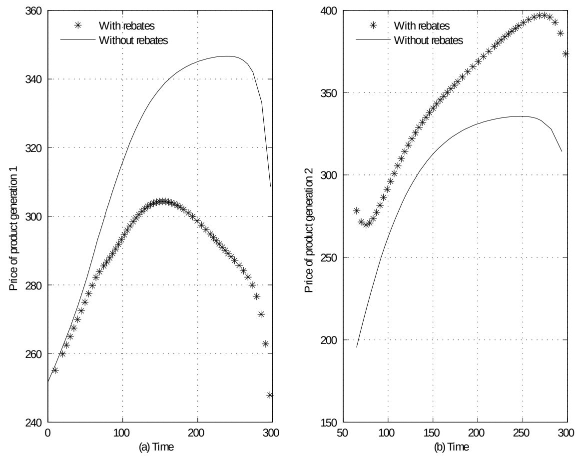


Figure 5. Effect of rebate program on optimal prices



Figures 4 and 5 illustrate the effect of the upgrade-rebate program on the diffusion of the fraction of buyers and the optimal prices. Comparing visually Figure 4 with the case when the prices are given exogenously (Figure 1) we observe that the diffusion acceleration effect of the program is less prominent. However, the implementation of the program allows the company to charge higher prices for the second product generation. On the other hand, the company charges lower prices for the first product generation so that to mitigate the deceleration effect of the program on the fraction of buyers of the first generation product (see Figure 5).

Overall, for this set of parameters the optimal total discounted profit is $\Pi = 16120$ for the benchmark case and $\Pi = 16820$ if the company implements the trade-in strategy. As expected, the values of optimal total discounted profits for both scenarios are higher than the corresponding values when the prices are given exogenously. The implementation of trade-in strategy increases the total discounted profits of the company by nearly 4.34%. Interestingly, the percentage gain in profits is now less than that of the case when the prices are exogenously given. The intuitive explanation is that by handling prices the company has more degrees of freedom to accelerate the flows in-and-between both generations, so the effect of an additional control (e.g., rebates) is smaller.

Two generation example, when returns-upgrades are for free

The results of the previous examples show that the upgrade-rebate program substantially increases the profits of the firm. The program accelerates the diffusion, but also increases the total number of the buyers of the second generation product. Thus, it is still questionable if the increase in profits of the firm is due to faster diffusion of the sales or merely to more buyers of the second generation product.

We explore this issue by considering a situation where the firm provides the upgraders the product of the new generation for free. Providing free upgrading is suboptimal for the firm. However, this allows us to see if the increase of profits is due to diffusion acceleration of the second generation, as the firm does not earn anything from those customers who upgrade.

The firm's optimization problem is similar to that given in the first example with exogenous prices with the exception that $D_{12} = P_2$. We also abstract from the pricing strategy of the firm, assuming the prices as given. Thus, the BVP for this scenario is given as in (3.4)-(3.10), substituting for D_{12} by P_2.

We compute the numerical solution using the same values for the parameters. Because of space limitations, we do not provide the diffusion paths of the fraction of buyers. However, we report that the diffusion acceleration effect on the second generation product is prominent. Obviously, the optimal life-time profits are lower for this suboptimal case than in the first example, $\Pi = 15371$ for the benchmark case and $\Pi = 15902$ when the upgrade-rebate strategy is implemented. Thus, even though the company would not earn anything from those customers who buy the first generation product and then upgrade to the second generation, it still faces a 3.45% increase in profits which is due to acceleration effect of the program.

3.3.2 Managing Three Generations

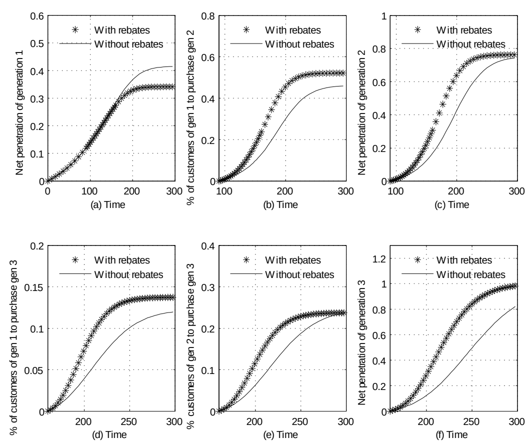
We also conduct a similar analysis for the case when the company manages diffusion waves of 3 generations of a product. We provide the representation of the optimal control problem and its maximum principle conditions in the Appendix A3.

We solve numerically the system of differential equations for similar values of parameters. We set the market size of the third generation product as $m_3 = 3000$. We take the following values for the coefficients of innovation: $p_{11} = 0.002$, $p_{12} = 0.002$, $p_{13} = 0.002$, $p_{22} = 0.002$, $p_{23} = 0.002$, $p_{33} = 0.004$; and coefficients of imitation: $q_{11} = 0.03$, $q_{12}^j = 0.03$, $q_{13}^j = 0.03$, $q_{22}^j = 0.05$, $q_{23}^j = 0.05$, $q_{33}^j = 0.05$. All the price sensitivity parameters are set to $\theta_{ij} = 0.002$. We assume that $b_{13}^j = 0.07$, $b_{23}^j = 0.07$. For the upgraders diffusion equation we set the parameters as $\gamma_{ij} = 0.05$, $\beta_{ij}^j = 0.1$, $\phi_{ij} = 0.0005$. Furthermore, the unit price and cost parameters of the third generation are assumed to be equal to those of the previous generations $P_3 = 300$ and $c_3 = 100$, respectively. We assume that the second and the third generations are introduced at $\tau_2 = 90$ and $\tau_3 = 160$. The rest of the parameters of the model are assumed to be the same as in the case of the two generations examples.

For this set of parameters the optimal total discounted profit is $\Pi = 8509$ for the benchmark case and $\Pi = 8331$ if the company implements the trade-in strategy. The optimal discounted profits under the two scenarios are less than in the case of two generations as innovator and imitator coefficients are smaller. The percentage gain in profits is also less, mostly because the second and the third generations are introduced later (at 90th and 160th period, respectively). Still, the implementation of trade-in strategy increases the total discounted profits of the company by nearly 2.13%.

Figure 6 shows the effect of the program on the diffusion of the buyers' proportions. On the one hand, the trade-in strategy slows down the diffusion of the proportion of the buyers of the first generation product (Figure 6 panel (a)). On the other hand, it speeds up the diffusion of the other proportions of the customers. However, this effect is different across the proportion of buyers of second and third generations. The second generation's diffusion is accelerated less as some fraction of customers of the second generation trade-up for the third generation, making the opposite effect on the acceleration of the second generation. The upgrade-rebate program mostly affects the proportions of buyers of the third generation (f_{13}, f_{23}, f_{33}).

Figure 6. Effect of rebate program on diffusion



3.4 Empirical Application

As mentioned in the Introduction, trade-in programs are implemented by companies, as well as by governments. This section presents an empirical application of the upgrade-rebate model to the automobile industry in Spain. In 1997 Spanish government launched an upgrade-rebate program with indefinite duration known as *Plan Prever*, endowing car upgrades with a fixed rebate of 480 euros (as a reduction in the new car registration tax) provided that their old car is deregistered and scrapped. In this study, we used monthly data from January 1970 till June 2000. The data set includes the number of Plan Prever scrapped old cars, the number of sold cars, and the average unit prices. The raw data were obtained from the Dirección General de Tráfico (Spanish administration department in charge of Motor Vehicles), Instituto de Estudios de Automoción (Institute for Research in the Automobile Industry of Spain, that is part of the National Carmakers Association ANFAC), and Infoadex (a Spanish market research company, http://www.infoadex.es). We compute a wholesale cars price index in Euros (1995 base). Since the data on scrapped cars are aggregated for the cars of the same age, we make a simplifying assumption that all the car models produced before 1997 are labelled as "first" generations, and those launched after the commence of the program as "second" generation. Also, we set the market potentials of the "first" and the "second" generations as $m_1 = 40000$ and $m_2 = 3000$ (in thousands), respectively which are reasonable assumptions validated by the opinion of industry analysts.

The first step is the model calibration. For these data, we compute f_{11}, f_{12}, and f_{22} as a fraction of cumulative monthly sales over the market potentials of both generations. Using sales data of the "first" generation from 1970 till 1997 and cumulative car sales at the beginning of the period , we first estimate p_{11}, q_{11}, and θ_{11} in equation (3.4), including an additional dummy variable to account for the recession years. In particular, in the "first" generation during the years 1970-1996, we consider the following model

$$\dot{f}_{11} = (p_{11} + q_{11}\ f_{11}(t))\,(1 - f_{11}(t))\,(1 - \theta_{11}\ P_1(t))\,(1 + a\ RC(t))\,,$$

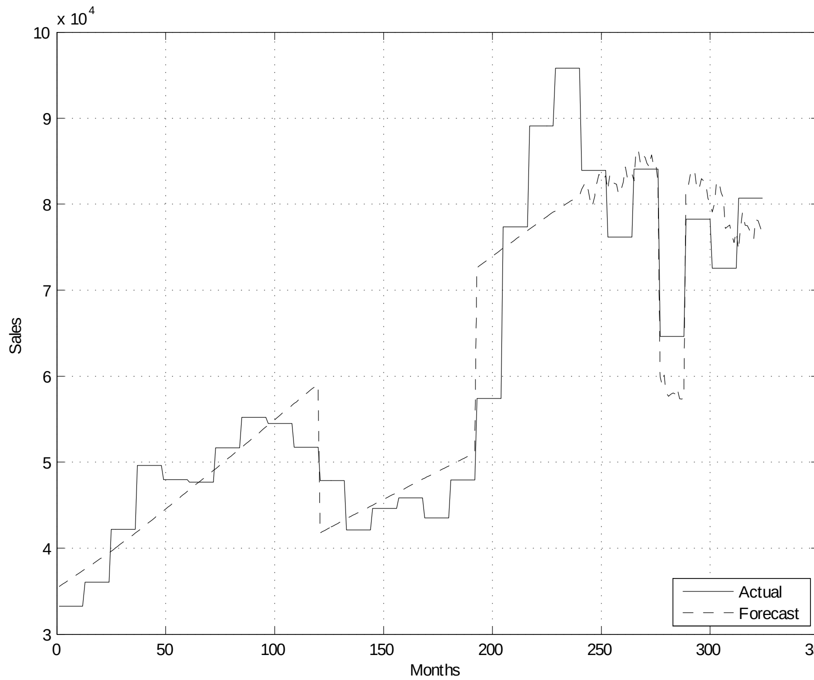
where RC is a dummy variable which takes value 1 in the recession years 1980, 1981, 1982, 1983, 1984, 1985 and 1993 and zero otherwise. The regression parameters were estimated by the ordinary least squares (OLS) method using STATA. The least-square regression coefficients can be found in Table 1. In the first column we report the coefficient estimates associated with each variable and the standard errors are reported in the second column. All the coefficient estimates are significantly different from zero, and the degree of explanation of this model is quite significant, as its adjusted R-Squared value is 0.99. Figure 7 shows that the model fits well enough the sales of the "first" generation.

Table 1. Parameter estimates for "first" generation. Dependent variable: cumulative car sales 90-97

Variables	Coefficients	Standard Error
p_{11}	.0024427**	.0005044
q_{11}	.0096211**	.0002366
θ_{11}	.0000385**	$3.23e-06$
a	$-.2950706$**	.0107121

(*) significant at the 10%, (**) 5% level

Figure 7. "First" generation, 1970-1996



Then, using the estimates given in Table 1 and the average price of the last month prior to introduction of Prever, we get the predicted sales of the "first" generation cars after 1997. Substructing from the predicted sales the total actual sales of "first" generation cars after 1997, we get the number of "second" generation cars that were bought monthly by the potential customers of the "first" generation (i.e. natural upgraders). Taking this number as a fraction over m_1, we obtain f_{12}. Substructing from the total number of sales of the "second" generation cars the number of upgraders and dividing this over market potential m_2, we get f_{22}. Finally, we obtain monthly r_{12} as a fraction of cumulative number of Prever program scrapped cars over cumulative number of "first" generation cars sales. As a result, the final version of the compiled data set covers the time period of January 1997 till June 2000 and contains all variables to estimate the system of non-linear equations given in (3.5) using nonlinear seemingly unrelated regression analysis.

Then, we first estimate

$$\dot{r_{12}}(t) = (\gamma + \beta\ r_{12}(t))(1 - r_{12}(t))\left(1 - \sum_{l=1}^{11} d_l m_l(t)\right),$$

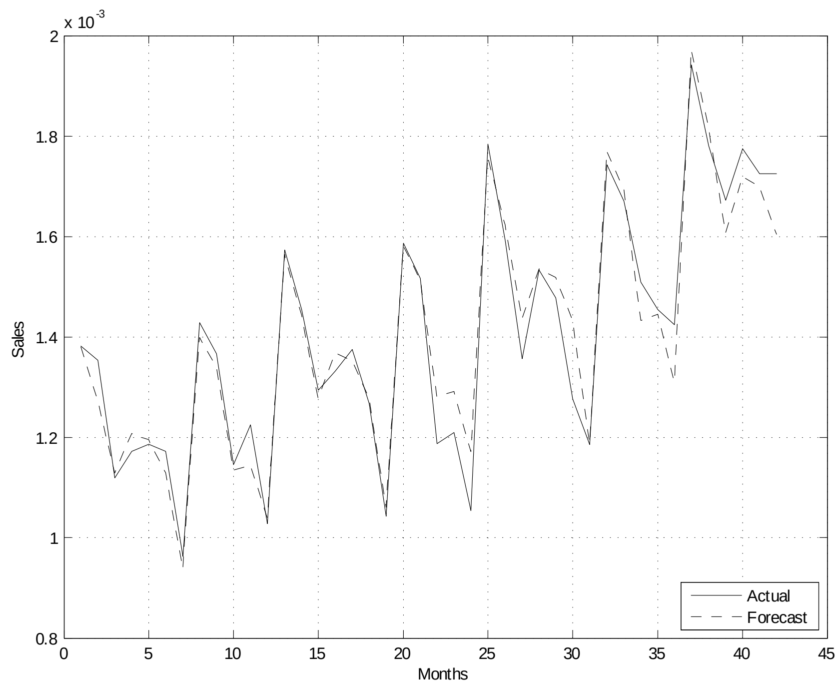
where $m_l(t)$ are seasonal (or monthly) dummy variables which take the value of one in a specific month and zero elsewhere. To avoid multicollinearity problems derived from the fact that $\sum_{l=1}^{12} m_l(t) = 1$, we have omitted one in the above specification. The regression parameters were estimated by the ordinary least squares (OLS) method using STATA. Its least-square regression coefficients can be found in Table 2. All the coefficient estimates are significantly different from zero, and the degree of explanation of this model is quite significant, as its adjusted R-Squared value is 0.99. Figure 8 shows the original time series against the forecasts of the fraction of individuals owning a product from the "first" generation who have upgraded it to a product of the "second" generation. We observe a good fit of the model with an increasing trend and a seasonality pattern of the rates, implying that Plan Prever generates incentives to upgrade the product to the "second" generation.

Table 2. Parameter estimates for scraped cars. Dependent variable: cumulative number of scrapped cars 97-00

Variables	Coefficients	Standard Error
γ	.0009249**	.000034
β	.0095854**	.0005886

(*) significant at the 10%, (**) 5% level

Figure 8. Upgraders



Assuming that the effect of scraped cars and prices is the same in both generations we have the following equations to estimate:

$$\dot{f}_{12} = (p_{12} + q_1 f_{12} + br_{12} f_{12})\,(1 - f_{11} - f_{12})(1 - \theta_2 P_2)\left(1 - \sum_{l=1}^{11} d_l m_l\right)$$
$$\dot{f}_{22} = (p_{22} + q_2 f_{12} + q_2 f_{22} + br_{12} f_{22})\,(1 - f_{22})(1 - \theta_2 P_2)\left(1 - \sum_{l=1}^{11} d_l m_l\right)$$

Here we again include some additional dummy variables to model the monthly pattern of sales. We estimate these system of nonlinear equations using STATA. Its least-square regression coefficients can be found in Table 3. All the coefficient estimates are significantly different from zero, and the degree of explanation of this model is quite significant, as its R-Squared value is 0.96.

Table 3. Parameter estimates for "first" generation. Dependent variable: cumulative car sales 90-97

Variables	Coefficients	Standard Error
p_{12}	.0094652**	.00317
q_1	-1.143466*	.65044
p_{22}	.0257948**	.0085261
q_2	.3049893**	.0848661
b	.6355303*	0.378241
θ_2	0.0000558**	.00000

(*) significant at the 10%, (**) 5% level

Figure 9 shows the original time series against the forecasts of the departing rate from potential customers of the "first" generation to purchase the "second" generation. Although this rate is more or less stable for the period of the study, the net penetration of generation "two" shows an increasing trend and a seasonality pattern of the rates (see Figure 10).

Figure 9. Customers from market of 1 to buy the 2nd

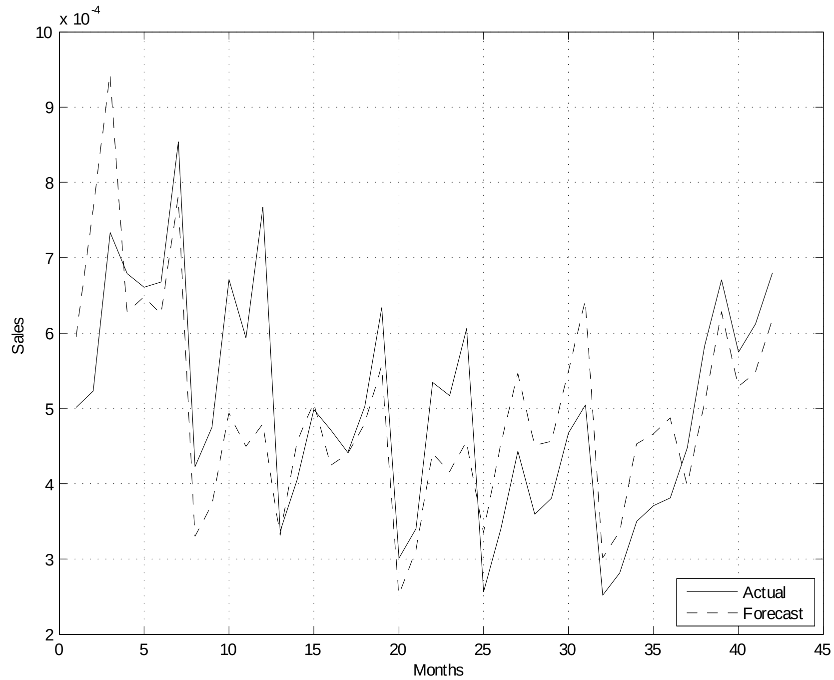
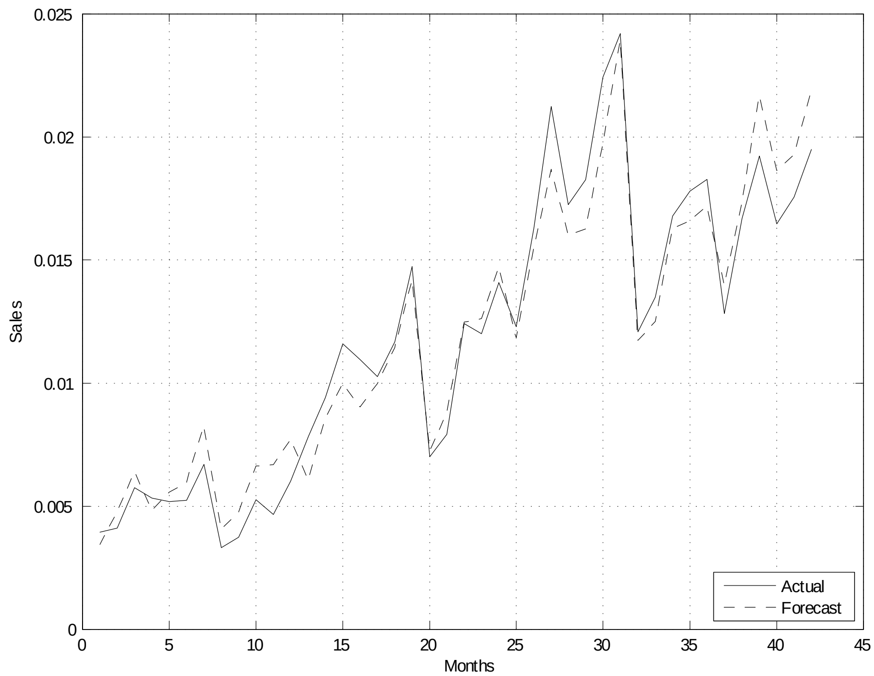


Figure 10. Net penetration of the 2nd



In this context, the automobile industry faces an optimal decision problem similar to the one described in Section 3.3.1, slightly modified to accommodate it to our data. As the rebate amount is fixed over time, we set $\phi_{12}D_{12}(t) = 4.8$ (in thousands), and assume that the decision maker maximizes its profits only by choosing the prices of the "first" and the "second" generations. To simplify the exposition, we set costs considering a 35% mark-up value for the "first" generation and 40% for the "second" (i.e. mark up = (price-cost)/price). Using the parameter estimates of the model obtained above, we compute the optimal pricing decisions for both the "first" and the "second" generations. For the optimal decisions, we consider a non recession "what if" scenario (i.e., we set the recession years dummies as zero). Figure 11 shows the optimal prices for the "first" and the "second" generations. Optimal price for the "first" generation increases slowly at first when the "first" generation models were just produced, before increasing sharply right after 1997 and then falling further. Interestingly, both optimal prices for the “first” and the "second" generations reach a peak almost at the same period of time. This suggests that in the absence of competition and scale economies, variable margins should increase over time.

Figure 11. Optimal prices

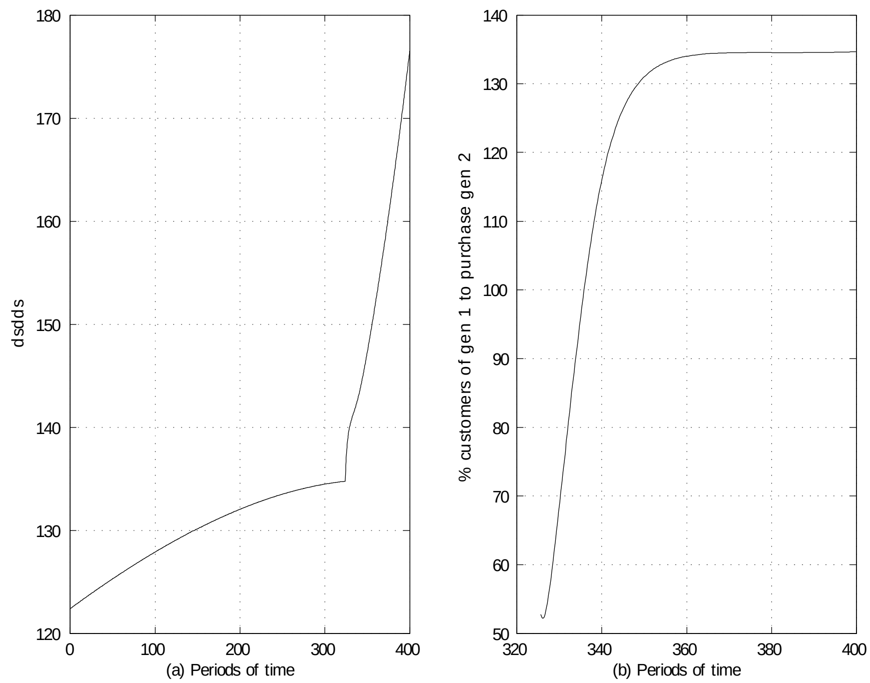
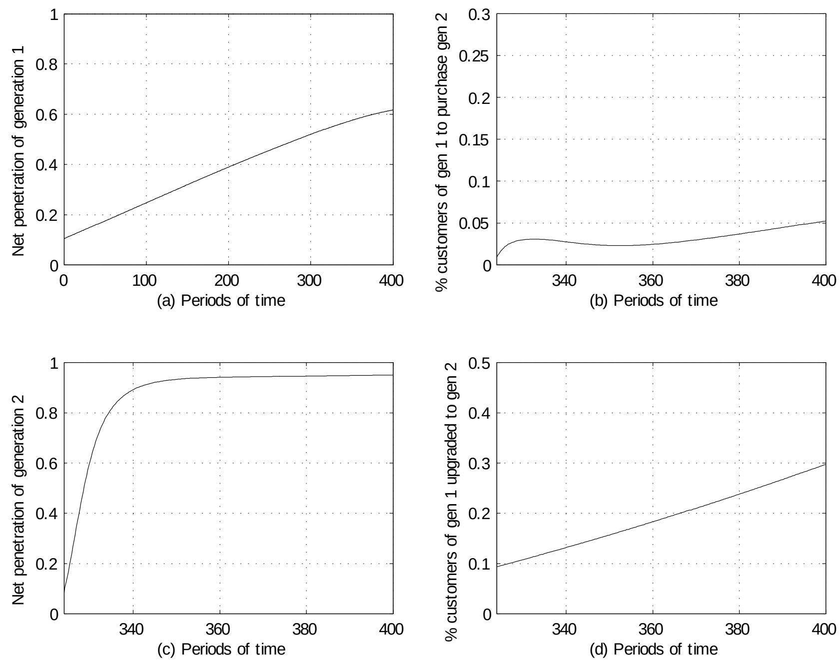


Figure 12 shows the diffusion of both product generations. The net diffusion of the “first” generation goes up slowly, while the net diffusion of the “second” increases abruptly and then remains steady. The percentage of buyers of the “first” generation that upgraded to the second one, is significantly higher that the percentage of potential customers of the “first” generation to purchase product of the second generation. This suggests that the use of upgrade-rebate program has been an optimal strategy in this context.

Figure 12. Diffusion of both generations



From the analysis of the results, we can conclude that this approach is able to analyze several generations, controlling with prices. We think that the industry might have improved revenues optimizing the rebate amount (complementing the government rebate), given the results of previous sections. Plan Prever expired in January 2008, given the Spanish government budgetary cuts introduced with the economic crisis starting the previous year. Interestingly, after the government withdrew the plan, some car manufacturers have launched their own upgrade-rebate programs. For example, in 2011 Peugeot Spain launched its own rebate program, with a 2.600 euros on average by the old car (that will be scrapped regardless of its value) for the last versions of the several models (the rebate changes with the model). Renault also introduced a trade-in program with 2500 euros rebate, and Citroën up to 300 Euros. In 2012, Nissan also launched its ownrebate for some models, and Ford did the same with a rebate up to 5000 Euros.

Clearly, the industry is intensively using this strategy to overcome the crisis downturn of that period. These programs are industry driven, and performance data are not publicly available.

We acknowledge that the presented application has some limitations. First, during the considered period the industry did not control for an optimal rebate discount, but simply accepted the government subsidy adapting their prices to this scenario which is why in this section we simplify the model assuming that the producer only controls for the prices. Second, before the Prever Plan, the Spanish government tested the effectiveness of similar car replacement promotions using temporary programs during the 1990s (Plan Renove I lasted 6 months, and Plan Renove II lasted 9 months). We are overlooking possible delayed effects of older temporal programs, given that the aim of this section is not a full analysis of this industry but to show the applicability of the model.

3.5 Conclusion

This paper studies how to use upgrade-rebate program as a strategic tool to manage the diffusion of successive product generations. Prior literature has considered upgrade-rebate program as a means to disable second hand market of the products, to decrease the regret of customers of buying the old generation product, to price discriminate the customers. However, most of the previous models analyze the program in an essentially static context. This paper takes a different approach to analyze the upgrade-rebate program in a dynamic context, where it is used to control the whole diffusion of the successive generations.

We first build a general diffusion model for successive product generations, and demonstrate conditions under which it is optimal to use upgrade-rebate programs. Our model allows for leapfroggers, and can be easily adopted to analyze the upgraders through the upgrade-rebate program. The model can be applied to study the diffusion of non-durables, as well as durable products.

In order to analyze the effect of upgrade-rebate strategy, we particularize the general model for some concrete examples and numerically solve them for certain sets of parameter values. For all the examples, we demonstrate that the trade-in program accelerates the diffusion of the later generations but has the reverse effect for the diffusion of the old generation product. In order to quantify the effect of the program, we compute the optimal discounted profits under scenarios of implementing upgrade-rebate strategy or not. We find that the implementation of the upgrade-rebate strategy provides a $2-5\%$ increase in total discounted profits.

The size of the percentage gain in profits varies depending on several conditions. For two generations, we demonstrate that when the firm takes the product prices as given, the gain in profits is the highest, more than 5%. However, when the firm also controls for prices, the increase in profits is less: 4.34%. We show that under trade-in strategy the firm changes its pricing policy, charging lower levels of prices of the first and higher levels for the second generation products.

We also find that for the firm it is optimal to provide the upgraders rebate amounts even higher than the price of the new generation up to some time. Subsidizing adopters and product giveaways have been studied previously in the context of one product generation (e.g. Kalish and Lilien 1983, Lehmann and Esteban-Bravo 2006). Kalish and Lilien (1983) demonstrate that mostly optimal subsidies are likely to decline over time. Lehmann and Esteban-Bravo (2006) examine when it is optimal to give away some products to innovators in a market consisting of two segments. In the context of two product generations, we show that even if the firm provides the upgraders the new generation products for free, it still faces more than a 3% increase in profits. We argue that this gain in profits is not due to more adopters of the new generation product nor to the change in pricing strategy of the firm, but merely to the diffusion acceleration effect of the upgrade-rebate strategy.

Most managers favor faster waves of their product generations. We show that a manager could create a tsunami through a trade-in strategy accelerating the diffusion of later generations. Obviously, this strategy should be carried out in parallel with other marketing mix decisions, such as pricing. A simpler implication is that a manager could provide new generation products to the owners of the old generation, even for free, in order to achieve rapid adoption of the new generation.

Our modeling framework has made a number of assumptions that can be relaxed to have a better understanding of trade-in strategy in a dynamic context. For example, we assumed that the proportion of customers who upgrade once never upgrade again in the future, i.e. rebate-upgrades are not iterated. We can relax this assumption, albeit increasing complexity. For instance, we can allow a maximum of J iterated rebate-upgrades, then sales for all generations are given by the vector

$$\begin{aligned} S(t) &= \dot{\mathbf{f}}(t)' m + \sum_{j=1}^{J} \frac{d}{dt}\left\{\mathbf{r}(t)^{j}\,\mathbf{f}(t)' m\right\}, \\ \frac{d}{dt}\left\{\mathbf{r}(t)^{j}\,\mathbf{f}(t)' m\right\} &= \left(j\mathbf{f}(s)\,\dot{\mathbf{r}}(s)^{j-1} + \dot{\mathbf{f}}(s)\,\mathbf{r}(s)^{j}\right)' m \end{aligned}$$

where $\frac{d}{dt}\left\{\mathbf{r}(t)^{j}\,\mathbf{f}(t)' m\right\}$ are sales associated to $j-th$ iterated upgrade, and the last equality is based on

$$\left(\mathbf{f}(t)\,\mathbf{r}(t)^{j}\right)' m = \int_0^t \left(j\mathbf{f}(s)\,\dot{\mathbf{r}}(s)^{j-1} + \dot{\mathbf{f}}(s)\,\mathbf{r}(s)^{j}\right)' mds.$$

since $j\int_0^t \left(\mathbf{f}(s)\,\dot{\mathbf{r}}(s)^{j-1}\right)' mds = \left(\mathbf{f}(t)\,\mathbf{r}(t)^{j}\right)' m - \int_0^t \left(\dot{\mathbf{f}}(s)\,\mathbf{r}(s)^{j}\right)' mds$. But we have not dealt with this case in the paper, i.e. we set $J = 1$.

Additional refinement could be considering other marketing mix variables such as advertising, and the timing decision of product generation launch and trade-in strategies. Neither do we consider competition for the firm from the second-hand markets. We leave these extensions for future research.

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3.7 Appendix

3.7.1 A1: Two generations with exogenous prices

Let $\lambda = (\lambda_1(t), \lambda_2(t), \lambda_3(t))$ be the multiplier vector associated with the state variables f_{11}, f_{12}, f_{22}, respectively, and μ be the multiplier associated with r_{12}. Also, let the Hamiltonian function H of the optimal control be given by:

$$H = \begin{cases} H_1 & \text{if } \tau_1 < t < \tau_2 \\ H_2 & \tau_2 < t < \infty \end{cases}$$

where

$$H_1(f_t, r_t, D_t, \lambda_t, \mu_t) = e^{-rt}\Pi_1(t) + \lambda_1 \dot{f}_{11}$$

and

$$H_2(f_t, r_t, D_t, \lambda_t, \mu_t) = e^{-rt}\Pi_2(t) + \lambda_1 \dot{f}_{11} + \lambda_2 \dot{f}_{12} + \lambda_3 \dot{f}_{22} + \mu_1 \dot{r}_{12}$$

where λ_t and μ_t be the multiplier matrixes associated with the state variables f_t and r_t, respectively. The maximum principle conditions are

$$\begin{aligned} \frac{\partial H_t}{\partial D_{12}} &= 0, \\ \dot{\lambda}_t &= -\frac{\partial H_t}{\partial f_t}, \\ \dot{\mu}_t &= -\frac{\partial H_t}{\partial r_t} \end{aligned}$$

together with initial conditions at introduction times, and transversality conditions of λ_t and μ_t tending to zero (numerically we equalize them to zero for large t). Also, we require the state variables and multipliers be continuous at τ_2.

Maximum Principle conditions for $\tau_1 < t < \tau_2$

As the second generation is not introduced by time τ_2, there is no optimality condition for the rebates. The multiplier is obtained from the following differential equation:

$$\begin{aligned} \dot{\lambda}_1 &= -e^{-rt}\left(q_{11}\left(1-f_{11}\left(t\right)\right)-p_{11}-q_{11}f_{11}\left(t\right)\right)\left(1-\theta_{11}P_1\left(t\right)\right)\left(m_1\left(P_1\left(t\right)-c_1\left(t\right)\right)+\lambda_1 e^{rt}\right) \\ \dot{\lambda}_2 &= 0, \dot{\lambda}_3 = 0, \dot{\mu}_1 = 0 \end{aligned} \tag{3.6}$$

Maximum Principle conditions $\tau_2 < t < T$

The optimality conditions for rebate D_{12} yield:

$$\begin{aligned} 0 &= e^{-rt} f_{11} m_1\left(\gamma_{12}+\beta_{12} r_{12}\left(t\right)\right)\left(1-r_{12}\left(t\right)\right)\phi_{12}\left(\left(P_2\left(t\right)-c_2\left(t\right)\right)-2D_{12}\left(t\right)+\frac{\mu_1 e^{rt}}{f_{11} m_1}\right) \\ &\quad -e^{-rt}\left(p_{11}+q_{11}f_{11}\left(t\right)\right)\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)\right)\left(1-\theta_{11}P_1\left(t\right)\right)m_1 r_{12} \end{aligned}$$

$$\begin{aligned} D_{12}\left(t\right) &= \frac{1}{2}\left(\left(P_2\left(t\right)-c_2\left(t\right)\right)+\frac{\mu_1 e^{rt}}{f_{11} m_1}\right) \\ &\quad -\frac{\left(p_{11}+q_{11}f_{11}\left(t\right)\right)\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)\right)\left(1-\theta_{11}P_1\left(t\right)\right)r_{12}}{2f_{11}\left(\gamma_{12}+\beta_{12}r_{12}\left(t\right)\right)\left(1-r_{12}\left(t\right)\right)\phi_{12}} \end{aligned}$$

The state variables and multipliers are obtained from the Boundary Value Problem, consisted of the four differential equations of the state variables and 4 differential equations of multipliers given by:

$$\begin{aligned} &\dot{\lambda}_1 = -e^{-rt}\left(q_{11}\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)\right)-p_{11}-q_{11}f_{11}\left(t\right)\right)\left(1-\theta_{11}P_1\left(t\right)\right)\times \\ &\quad \left(m_1\left(P_1\left(t\right)-c_1\left(t\right)+r_{12}\left(P_2\left(t\right)-c_2\left(t\right)-D_{12}\left(t\right)\right)\right)+\lambda_1 e^{rt}\right)- \\ &\quad e^{-rt}\left(b^1_{12}r_{12}\left(t\right)\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)\right)-\left(p_{12}+q^1_{12}f_{12}\left(t\right)+q^1_{22}f_{22}\left(t\right)+b^1_{12}r_{12}\left(t\right)f_{11}\left(t\right)\right)\right) \\ &\quad \times\left(1-\theta_{12}P_2\left(t\right)\right)\left(m_1\left(P_2\left(t\right)-c_2\left(t\right)\right)+\lambda_2 e^{rt}\right)- \\ &\quad e^{-rt}b^2_{12}r_{12}\left(t\right)\left(1-f_{22}\left(t\right)\right)\left(1-\theta_{22}P_2\left(t\right)\right)\left(m_2\left(P_2\left(t\right)-c_2\left(t\right)\right)+\lambda_3 e^{rt}\right)- \\ &\quad e^{-rt}m_1\left(\gamma_{12}+\beta_{12}r_{12}\left(t\right)\right)\left(1-r_{12}\left(t\right)\right)\phi_{12}D_{12}\left(t\right)\left(P_2\left(t\right)-c_2\left(t\right)-D_{12}\left(t\right)\right) \end{aligned} \tag{3.7}$$

$$\begin{aligned}
\dot{\lambda}_2 &= e^{-rt}\left(p_{11}+q_{11}f_{11}\left(t\right)\right)\left(1-\theta_{11}P_1\left(t\right)\right)\times \\
&\left(m_1\left(P_1\left(t\right)-c_1\left(t\right)+r_{12}\left(P_2\left(t\right)-c_2\left(t\right)-D_{12}\left(t\right)\right)\right)+\lambda_1 e^{rt}\right)- \\
&e^{-rt}\left(q_{12}^1\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)\right)-\left(p_{12}+q_{12}^1 f_{12}\left(t\right)+q_{22}^1 f_{22}\left(t\right)+b_{12}^1 r_{12}\left(t\right)f_{11}\left(t\right)\right)\right)\times \\
&\left(1-\theta_{12}P_2\left(t\right)\right)\left(m_1\left(P_2\left(t\right)-c_2\left(t\right)\right)+\lambda_2 e^{rt}\right)- \\
&e^{-rt}q_{12}^2\left(1-f_{22}\left(t\right)\right)\left(1-\theta_{22}P_2\left(t\right)\right)\left(m_2\left(P_2\left(t\right)-c_2\left(t\right)\right)+\lambda_3 e^{rt}\right)
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\dot{\lambda}_3 &= 0- \\
&e^{-rt}q_{22}^1\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)\right)\left(1-\theta_{12}P_2\left(t\right)\right)\left(m_1\left(P_2\left(t\right)-c_2\left(t\right)\right)+\lambda_2 e^{rt}\right)- \\
&e^{-rt}\left(q_{22}^2\left(1-f_{22}\left(t\right)\right)-\left(p_{22}+q_{12}^2 f_{12}\left(t\right)+q_{22}^2 f_{22}\left(t\right)+b_{12}^2 r_{12}\left(t\right)f_{11}\left(t\right)\right)\right)\times \\
&\left(1-\theta_{22}P_2\left(t\right)\right)\left(m_2\left(P_2\left(t\right)-c_2\left(t\right)\right)+\lambda_3 e^{rt}\right)
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\dot{\mu}_1 &= -e^{-rt}\left(\beta_{12}\left(1-r_{12}\left(t\right)\right)-\left(\gamma_{12}+\beta_{12}r_{12}\left(t\right)\right)\right)\phi_{12}D_{12}\left(t\right)\times \\
&\left(f_{11}m_1\left(P_2\left(t\right)-c_2\left(t\right)-D_{12}\left(t\right)\right)+\mu_1 e^{rt}\right)- \\
&e^{-rt}\left(p_{11}+q_{11}f_{11}\left(t\right)\right)\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)\right)\left(1-\theta_{11}P_1\left(t\right)\right)m_1\left(P_2-c_2-D_{12}\left(t\right)\right) \\
&-e^{-rt}b_{12}^1 f_{11}\left(t\right)\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)\right)\left(1-\theta_{12}P_2\left(t\right)\right)\left(m_1\left(P_2\left(t\right)-c_2\left(t\right)\right)+\lambda_2 e^{rt}\right) \\
&-e^{-rt}b_{12}^2 f_{11}\left(t\right)\left(1-f_{22}\left(t\right)\right)\left(1-\theta_{22}P_2\left(t\right)\right)\left(m_2\left(P_2\left(t\right)-c_2\left(t\right)\right)+\lambda_3 e^{rt}\right)
\end{aligned} \tag{3.10}$$

For the benchmark scenario when the upgrade-rebate program is not implemented, the optimal control problem is a special case where

$$D_{12}(t) = 0, \dot{r}_{12} = 0, \dot{\mu}_1 = 0$$

3.7.2 A.2. Two generations with endogenous prices

The optimal solution of this scenario is similar to that of the case when the firm takes the prices as given. The maximum principle conditions are

$$\begin{aligned}
\frac{\partial H_t}{\partial D_{12}} &= 0, \\
\frac{\partial H_t}{\partial P_1} &= 0, \\
\frac{\partial H_t}{\partial P_2} &= 0, \\
\dot{\lambda}_t &= -\frac{\partial H_t}{\partial f_t}, \\
\dot{\mu}_t &= -\frac{\partial H_t}{\partial r_t}
\end{aligned}$$

together with initial conditions at introduction times, and transversality conditions λ_t and μ_t tending to zero.

Maximum Principle conditions $\tau_1 < t < \tau_2$

The optimality condition for price is:

$$0 = -e^{-rt}(p_{11} + q_{11} f_{11}(t))(1 - f_{11}(t))\,\theta_{11} m_1 \left(P_1(t) - c_1(t) - \frac{(1 - \theta_{11} P_1(t))}{\theta_{11}} + \frac{\lambda_1}{e^{-rt} m_1} \right)$$

We rearrange the terms to get:

$$P_1(t) = \frac{1}{2}\left(c_1(t) + \frac{1}{\theta_{11}} - \frac{\lambda_1}{e^{-rt} m_1} \right)$$

As the second generation is not introduced before τ_2, there is no optimality condition for rebate and price of the second generation. The multiplier is given from the differential equation as in (6).

Maximum Principle conditions $\tau_2 < t < T$

The optimality conditions for discount D_{12} yield:

$$\begin{aligned} 0 &= e^{-rt} f_{11} m_1 \left(\gamma_{12} + \beta_{12} r_{12}(t)\right)\left(1 - r_{12}(t)\right)\phi_{12}\left(\left(P_2(t) - c_2(t)\right) - 2D_{12}(t) + \frac{\mu_1 e^{rt}}{f_{11} m_1}\right) \\ &\quad - e^{-rt}\left(p_{11} + q_{11} f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right)\left(1 - \theta_{11} P_1(t)\right) m_1 r_{12} \end{aligned}$$

The optimality conditions for prices are:

$$\begin{aligned} 0 &= -e^{-rt}\left(p_{11} + q_{11} f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right)\theta_{11} m_1 \times \\ &\quad \left(P_1(t) - c_1(t) + r_{12}\left(P_2(t) - c_2(t)\right) - \frac{\left(1 - \theta_{11} P_1(t)\right)}{\theta_{11}} - r_{12} D_{12}(t) + \frac{\lambda_1}{e^{-rt} m_1}\right) \end{aligned}$$

$$\begin{aligned} 0 &= -e^{-rt}\left(p_{12} + q^1_{12} f_{12}(t) + q^1_{22} f_{22}(t) + b^1_{12} r_{12}(t) f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right)\theta_{12} m_1 \times \\ &\quad \left(P_2(t) - c_2(t) - \frac{\left(1 - \theta_{12} P_2(t)\right)}{\theta_{12}} + \frac{\lambda_2}{e^{-rt} m_1}\right) - \\ &\quad - e^{-rt}\left(p_{22} + q^2_{12} f_{12}(t) + q^2_{22} f_{22}(t) + b^2_{12} r_{12}(t) f_{11}(t)\right)\theta_{22} m_2 \times \\ &\quad \left(P_2(t) - c_2(t) - \frac{\left(1 - \theta_{22} P_2(t)\right)}{\theta_{22}} + \frac{\lambda_3}{e^{-rt} m_2}\right) \\ &\quad e^{-rt} f_{11} m_1 \left(\gamma_{12} + \beta_{12} r_{12}(t)\right)\left(1 - r_{12}(t)\right)\phi_{12} D_{12}(t) \\ &\quad e^{-rt}\left(p_{11} + q_{11} f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right)\left(1 - \theta_{11} P_1(t)\right) m_1 r_{12}. \end{aligned}$$

We simplify the equations and represent the system as above, given three equations in matrix form as: $\Gamma(P_1(t), P_2(t), D_{12}(t))' = \Psi$, where

$$\begin{aligned}
\Gamma_{11} &= \left(p_{11} + q_{11} f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right)\theta_{11} r_{12} \\
\Gamma_{12} &= f_{11}\left(\gamma_{12} + \beta_{12} r_{12}(t)\right)\left(1 - r_{12}(t)\right)\phi_{12} \\
\Gamma_{13} &= -2 f_{11}\left(\gamma_{12} + \beta_{12} r_{12}(t)\right)\left(1 - r_{12}(t)\right)\phi_{12} \\
\Psi_1 &= -\Bigg[f_{11}\left(\gamma_{12} + \beta_{12} r_{12}(t)\right)\left(1 - r_{12}(t)\right)\phi_{12}\left(-c_2(t) + \frac{\mu_1 e^{rt}}{f_{11} m_1}\right) \\
&\quad - \left(p_{11} + q_{11} f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right) r_{12}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\Gamma_{21} &= 2 \\
\Gamma_{22} &= r_{12} \\
\Gamma_{23} &= -r_{12} \\
\Psi_2 &= -\left[-c_1(t) - r_{12} c_2(t) - \frac{1}{\theta_{11}} + \frac{\lambda_1}{e^{-rt} m_1}\right]
\end{aligned}$$

$$\begin{aligned}
\Gamma_{31} &= -\left(p_{11} + q_{11} f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right)\theta_{11} m_1 r_{12} \\
\Gamma_{32} &= -2\left(p_{12} + q_{12}^1 f_{12}(t) + q_{22}^1 f_{22}(t) + b_{12}^1 r_{12}(t) f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right)\theta_{12} m_1 \\
&\quad -2\left(p_{22} + q_{12}^2 f_{12}(t) + q_{22}^2 f_{22}(t) + b_{12}^2 r_{12}(t) f_{11}(t)\right)\left(1 - f_{22}(t)\right)\theta_{22} m_2 \\
\Gamma_{33} &= f_{11} m_1\left(\gamma_{12} + \beta_{12} r_{12}(t)\right)\left(1 - r_{12}(t)\right)\phi_{12} \\
\Psi_3 &= \Big[\left(p_{12} + q_{12}^1 f_{12}(t) + q_{22}^1 f_{22}(t) + b_{12}^1 r_{12}(t) f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right)\times \\
&\quad \theta_{12} m_1\left(-c_2(t) - \frac{1}{\theta_{12}} + \frac{\lambda_2}{e^{-rt} m_1}\right) + \\
&\quad \left(p_{22} + q_{12}^2 f_{12}(t) + q_{22}^2 f_{22}(t) + b_{12}^2 r_{12}(t) f_{11}(t)\right)\left(1 - f_{22}(t)\right)\times \\
&\quad \theta_{22} m_2\left(-c_2(t) - \frac{1}{\theta_{22}} + \frac{\lambda_3}{e^{-rt} m_2}\right) - \\
&\quad \left(p_{11} + q_{11} f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t)\right) m_1 r_{12}\Big]
\end{aligned}$$

Using Cramer's rule we express $P_1(t), P_2(t), D_{12}(t)$ in terms of the state variables and

multipliers. Finally, the optimal state variables and multipliers are obtained solving the Boundary Value Problem analogously to the model with exogenous prices.

For the benchmark scenario when the upgrade-rebate program is not implemented, the optimal control problem is a special case where

$$\dot{r}_{12} = 0, D_{12} = 0, \dot{\mu}_1 = 0$$

3.7.3 A.3. Three generations

We provide the solution for the case of 3 generations only for when the 3rd generation is introduced (i.e. $t > \tau_3$). The optimality conditions before the introduction of the 3rd generation are similar to the case of two generations.

For the case of 3 generations, the half-vectorization of $\dot{\mathbf{f}}(t)$ and $\dot{\mathbf{r}}(t)$ matrices is the following:

$$\begin{cases} \dot{f}_{11} = (p_{11} + q_{11} f_{11}(t))(1 - f_{11}(t) - f_{12}(t) - f_{13}(t))(1 - \theta_{11} P_1(t)), \\ \dot{f}_{12} = (p_{12} + q_{12}^1 f_{12}(t) + q_{22}^1 f_{22}(t) + b_{12}^1 r_{12}(t) f_{11}(t))(1 - f_{11}(t) - f_{12}(t) - f_{13}(t)) \times \\ \qquad (1 - \theta_{12} P_2(t)) \\ \dot{f}_{13} = (p_{13} + q_{13}^1 f_{13}(t) + q_{23}^1 f_{23}(t) + q_{33}^1 f_{33}(t) + b_{13}^1 r_{13}(t) f_{11}(t) + \\ \qquad b_{23}^1 r_{23}(t)(f_{12}(t) + f_{22}(t)))(1 - f_{11}(t) - f_{12}(t) - f_{13}(t))(1 - \theta_{13} P_3(t)) \\ \dot{f}_{22} = (p_{22} + q_{12}^2 f_{12}(t) + q_{22}^2 f_{22}(t) + b_{12}^2 r_{12}(t) f_{11}(t))(1 - f_{22}(t) - f_{23}(t))(1 - \theta_{22} P_2(t)) \\ \dot{f}_{23} = (p_{23} + q_{13}^2 f_{13}(t) + q_{23}^2 f_{23}(t) + q_{33}^2 f_{33}(t) + b_{13}^2 r_{13}(t) f_{11}(t) + \\ \qquad b_{23} r_{23}(t)(f_{12}(t) + f_{22}(t)))(1 - f_{22}(t) - f_{23}(t))(1 - \theta_{23} P_3(t)) \\ \dot{f}_{33} = (p_{33} + q_{13}^3 f_{13}(t) + q_{23}^3 f_{23}(t) + q_{33}^3 f_{33}(t) + b_{13}^3 r_{13}(t) f_{11}(t) + \\ \qquad b_{23}^3 r_{23}(t)(f_{12}(t) + f_{22}(t)))(1 - f_{33}(t))(1 - \theta_{33} P_3(t)) \end{cases}$$

and

$$\begin{cases} \dot{r}_{12} = \left(\gamma_{12} + \beta_{12}^{1} r_{12}(t)\right)\left(1 - r_{12}(t) - r_{13}(t)\right)\phi_{12} D_{12}(t), \\ \dot{r}_{13} = \left(\gamma_{13} + \beta_{13}^{1} r_{13}(t) + \beta_{23}^{1} r_{23}(t)\right)\left(1 - r_{12}(t) - r_{13}(t)\right)\phi_{13} D_{13}(t), \\ \dot{r}_{23} = \left(\gamma_{23} + \beta_{13}^{2} r_{13}(t) + \beta_{23}^{2} r_{23}(t)\right)\left(1 - r_{23}(t)\right)\phi_{23} D_{23}(t) \end{cases}$$

The company profits are:

$$\begin{aligned} \Pi &= \int_0^\infty e^{-rt}\Pi(t)\,dt = \int_0^\infty e^{-rt}\Bigg(\dot{f}_{11} m_1 \left(P_1(t) - c_1(t)\right) \\ &\quad + \left(\dot{f}_{12} m_1 + \dot{f}_{22} m_2 + f_{11} m_1 \dot{r}_{12} + \dot{f}_{11} m_1 r_{12}\right)\left(P_2(t) - c_2(t)\right) \\ &\quad + \Bigg(\dot{f}_{13} m_1 + \dot{f}_{23} m_2 + \dot{f}_{33} m_3 + f_{11} m_1 \dot{r}_{13} + \dot{f}_{11} m_1 r_{13} \\ &\quad + \left(f_{12} m_1 + f_{22} m_2\right)\dot{r}_{23} + \left(\dot{f}_{12} m_1 + \dot{f}_{22} m_2\right) r_{23}\Bigg)\left(P_3(t) - c_3(t)\right) \\ &\quad - \left(f_{11} m_1 \dot{r}_{12} + \dot{f}_{11} m_1 r_{12}\right) D_{12}(t) - \left(f_{11} m_1 \dot{r}_{13} + \dot{f}_{11} m_1 r_{13}\right) D_{13}(t) \\ &\quad - \left(\left(f_{12} m_1 + f_{22} m_2\right)\dot{r}_{23} + \left(\dot{f}_{12} m_1 + \dot{f}_{22} m_2\right) r_{23}\right) D_{23}(t)\Bigg)\,dt \end{aligned}$$

Let the Hamiltonian function H of the optimal control be given by:

$$\begin{aligned} H_t\left(f_t, r_t, P_t, D_t, \lambda_t, \mu_t\right) &= e^{-rt}\Pi(t) + \lambda_1 \dot{f}_{11} + \lambda_2 \dot{f}_{12} + \lambda_3 \dot{f}_{13} + \lambda_4 \dot{f}_{22} + \lambda_5 \dot{f}_{23} + \lambda_6 \dot{f}_{33} \\ &\quad + \mu_1 \dot{r}_{12} + \mu_2 \dot{r}_{13} + \mu_3 \dot{r}_{23} \end{aligned}$$

where λ_t and μ_t be the multiplier matrices associated with the state variables f_t and r_t, respectively. The maximum principle conditions are

$$
\begin{aligned}
\frac{\partial H_t}{\partial D_{12}} &= 0, \frac{\partial H_t}{\partial D_{13}} = 0, \frac{\partial H_t}{\partial D_{23}} = 0 \\
\dot{\lambda}_t &= -\frac{\partial H_t}{\partial f_t}, \\
\dot{\mu}_t &= -\frac{\partial H_t}{\partial r_t}
\end{aligned}
$$

The optimality conditions for discounts D_{12},D_{13},D_{23} yield, respectively:

$$
\begin{aligned}
&e^{-rt} f_{11} m_1 \left(\gamma_{12} + \beta_{12}^1 r_{12}(t)\right) \left(1 - r_{12}(t) - r_{13}(t)\right) \phi_{12} \left(\left(P_2(t) - c_2(t)\right) - 2D_{12}(t) + \frac{\mu_1 e^{rt}}{f_{11} m_1} \right) \\
&- e^{-rt} \left(p_{11} + q_{11} f_{11}(t)\right) \left(1 - f_{11}(t) - f_{12}(t) - f_{13}(t)\right) \left(1 - \theta_{11} P_1(t)\right) m_1 r_{12} = 0
\end{aligned}
$$

$$
\begin{aligned}
D_{12}(t) &= \frac{1}{2} \left(\left(P_2(t) - c_2(t)\right) + \frac{\mu_1 e^{rt}}{f_{11} m_1} \right) \\
&\quad - \frac{\left(p_{11} + q_{11} f_{11}(t)\right) \left(1 - f_{11}(t) - f_{12}(t) - f_{13}(t)\right) \left(1 - \theta_{11} P_1(t)\right) r_{12}}{2 f_{11} \left(\gamma_{12} + \beta_{12}^1 r_{12}(t)\right) \left(1 - r_{12}(t) - r_{13}(t)\right) \phi_{12}}
\end{aligned}
$$

$$
\begin{aligned}
0 &= e^{-rt} f_{11} m_1 \left(\gamma_{13} + \beta_{13}^1 r_{13}(t) + \beta_{23}^1 r_{23}(t)\right) \left(1 - r_{12}(t) - r_{13}(t)\right) \phi_{13} \times \\
&\quad \left(\left(P_3(t) - c_3(t)\right) - 2D_{13}(t) + \frac{\mu_2 e^{-rt}}{f_{11} m_1} \right) \\
&\quad - e^{-rt} \left(p_{11} + q_{11} f_{11}(t)\right) \left(1 - f_{11}(t) - f_{12}(t) - f_{13}(t)\right) \left(1 - \theta_{11} P_1(t)\right) m_1 r_{13}
\end{aligned}
$$

$$
\begin{aligned}
D_{13}(t) &= \frac{1}{2} \left(\left(P_3(t) - c_3(t)\right) + \frac{\mu_2 e^{rt}}{f_{11} m_1} \right) \\
&\quad - \frac{\left(p_{11} + q_{11} f_{11}(t)\right) \left(1 - f_{11}(t) - f_{12}(t) - f_{13}(t)\right) \left(1 - \theta_{11} P_1(t)\right) r_{13}}{2 f_{11} \left(\gamma_{13} + \beta_{13}^1 r_{13}(t) + \beta_{23}^1 r_{23}(t)\right) \left(1 - r_{12}(t) - r_{13}(t)\right) \phi_{13}}
\end{aligned}
$$

$$
\begin{aligned}
0 &= e^{-rt}\left(f_{12}m_1 + f_{22}m_2\right)\left(\gamma_{23} + \beta_{13}^2 r_{13}(t) + \beta_{23}^2 r_{23}(t)\right)\left(1 - r_{23}(t)\right)\phi_{23} \times \\
&\left(\left(P_3(t) - c_3(t)\right) - 2D_{23}(t) + \frac{\mu_3 e^{rt}}{\left(f_{12}m_1 + f_{22}m_2\right)}\right) - \\
&e^{-rt}\left(p_{12} + q_{12}^1 f_{12}(t) + q_{22}^1 f_{22}(t) + b_{12}^1 r_{12}(t) f_{11}(t)\right)\left(1 - f_{11}(t) - f_{12}(t) - f_{13}(t)\right) \times \\
&\left(1 - \theta_{12}P_2(t)\right) m_1 r_{23}(t) - \\
&e^{-rt}\left(p_{22} + q_{12}^2 f_{12}(t) + q_{22}^2 f_{22}(t) + b_{12}^2 r_{12}(t) f_{11}(t)\right)\left(1 - f_{22}(t) - f_{23}(t)\right) \times \\
&\left(1 - \theta_{22}P_2(t)\right) m_2 r_{23}(t)
\end{aligned}
$$

$$
\begin{aligned}
D_{23}(t) &= \frac{1}{2}\left(\left(P_3(t) - c_3(t)\right) + \frac{\mu_3 e^{rt}}{\left(f_{12}m_1 + f_{22}m_2\right)}\right) \\
&\frac{-\left(p_{12}+q_{12}^1 f_{12}(t)+q_{22}^1 f_{22}(t)+b_{12}^1 r_{12}(t) f_{11}(t)\right)(1-f_{11}(t)-f_{12}(t)-f_{13}(t))(1-\theta_{12}P_2(t))m_1 r_{23}(t)}{2(f_{12}m_1+f_{22}m_2)\left(\gamma_{23}+\beta_{13}^2 r_{13}(t)+\beta_{23}^2 r_{23}(t)\right)(1-r_{23}(t))\phi_{23}} \\
&\frac{-\left(p_{22}+q_{12}^2 f_{12}(t)+q_{22}^2 f_{22}(t)+b_{12}^2 r_{12}(t) f_{11}(t)\right)(1-f_{22}(t)-f_{23}(t))(1-\theta_{22}P_2(t))m_2 r_{23}(t)}{2(f_{12}m_1+f_{22}m_2)\left(\gamma_{23}+\beta_{13}^2 r_{13}(t)+\beta_{23}^2 r_{23}(t)\right)(1-r_{23}(t))\phi_{23}}
\end{aligned}
$$

The state variables and multipliers are obtained from the following Boundary Value Problem:

$$
\begin{aligned}
\dot{\lambda}_1 \;=\; & -e^{-rt}\left(q_{11}\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)-f_{13}\left(t\right)\right)-p_{11}-q_{11}f_{11}\left(t\right)\right)\left(1-\theta_{11}P_1\left(t\right)\right)\times \\
& \left(m_1\left(P_1\left(t\right)-c_1\left(t\right)+r_{12}\left(P_2\left(t\right)-c_2\left(t\right)-D_{12}\left(t\right)\right)+r_{13}\left(P_3\left(t\right)-c_3\left(t\right)-D_{13}\left(t\right)\right)\right)+\lambda_1e^{rt}\right) \\
& -e^{-rt}\left(b_{12}^1r_{12}\left(t\right)\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)-f_{13}\left(t\right)\right)-\right. \\
& \left.\left(p_{12}+q_{12}^1f_{12}\left(t\right)+q_{22}^1f_{22}\left(t\right)+b_{12}^1r_{12}\left(t\right)f_{11}\left(t\right)\right)\right)\times \\
& \left(1-\theta_{12}P_2\left(t\right)\right)\left(m_1\left(P_2\left(t\right)-c_2\left(t\right)+r_{23}\left(P_3\left(t\right)-c_3\left(t\right)-D_{23}\left(t\right)\right)\right)+\lambda_2e^{rt}\right) \\
& -e^{-rt}\left(b_{13}^1r_{13}\left(t\right)\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)-f_{13}\left(t\right)\right)-\right. \\
& \left.\left(p_{13}+q_{13}^1f_{13}\left(t\right)+q_{23}^1f_{23}\left(t\right)+q_{33}^1f_{33}\left(t\right)+b_{13}^1r_{13}\left(t\right)f_{11}\left(t\right)+b_{23}^1r_{23}\left(t\right)\left(f_{12}\left(t\right)+f_{22}\left(t\right)\right)\right)\right)\times \\
& \left(1-\theta_{13}P_3\left(t\right)\right)\left(m_1\left(P_3\left(t\right)-c_3\left(t\right)\right)+\lambda_3e^{rt}\right) \\
& -e^{-rt}b_{12}^2r_{12}\left(t\right)\left(1-f_{22}\left(t\right)-f_{23}\left(t\right)\right)\left(1-\theta_{22}P_2\left(t\right)\right)\times \\
& \left(m_2\left(P_2\left(t\right)-c_2\left(t\right)+r_{23}\left(P_3\left(t\right)-c_3\left(t\right)-D_{23}\left(t\right)\right)\right)+\lambda_4e^{rt}\right) \\
& -e^{-rt}b_{13}^2r_{13}\left(t\right)\left(1-f_{22}\left(t\right)-f_{23}\left(t\right)\right)\left(1-\theta_{23}P_3\left(t\right)\right)\left(m_2\left(P_3\left(t\right)-c_3\left(t\right)\right)+\lambda_5e^{rt}\right) \\
& -e^{-rt}b_{13}^3r_{13}\left(t\right)\left(1-f_{33}\left(t\right)\right)\left(1-\theta_{33}P_3\left(t\right)\right)\left(m_3\left(P_3\left(t\right)-c_3\left(t\right)\right)+\lambda_6e^{rt}\right) \\
& -e^{-rt}m_1\left(\gamma_{12}+\beta_{12}^1r_{12}\left(t\right)\right)\left(1-r_{12}\left(t\right)-r_{13}\left(t\right)\right)\phi_{12}D_{12}\left(t\right)\left(P_2\left(t\right)-c_2\left(t\right)-D_{12}\left(t\right)\right) \\
& -e^{-rt}m_1\left(\gamma_{13}+\beta_{13}^1r_{13}\left(t\right)+\beta_{23}^1r_{23}\left(t\right)\right)\left(1-r_{12}\left(t\right)-r_{13}\left(t\right)\right)\times \\
& \phi_{13}D_{13}\left(t\right)\left(P_3\left(t\right)-c_3\left(t\right)-D_{13}\left(t\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_2 \;=\;& e^{-rt}\left(p_{11}+q_{11}f_{11}(t)\right)\left(1-\theta_{11}P_1(t)\right)\times \\
&\big(m_1\left(P_1(t)-c_1(t)+r_{12}\left(P_2(t)-c_2(t)-D_{12}(t)\right)+r_{13}\left(P_3(t)-c_3(t)-D_{13}(t)\right)\right)+\lambda_1 e^{rt}\big) \\
&-e^{-rt}\big(q_{12}^1\left(1-f_{11}(t)-f_{12}(t)-f_{13}(t)\right)-\big(p_{12}+q_{12}^1 f_{12}(t)+q_{22}^1 f_{22}(t)+b_{12}^1 r_{12}(t)\, f_{11}(t)\big)\big) \\
&\times\left(1-\theta_{12}P_2(t)\right)\big(m_1\left(P_2(t)-c_2(t)+r_{23}\left(P_3(t)-c_3(t)-D_{13}(t)\right)\right)+\lambda_2 e^{rt}\big) \\
&-e^{-rt}\big(b_{23}^1 r_{23}(t)\left(1-f_{11}(t)-f_{12}(t)-f_{13}(t)\right)- \\
&\big(p_{13}+q_{13}^1 f_{13}(t)+q_{23}^1 f_{23}(t)+q_{33}^1 f_{33}(t)+b_{13}^1 r_{13}(t)\, f_{11}(t)+b_{23}^1 r_{23}(t)\left(f_{12}(t)+f_{22}(t)\right)\big)\big) \\
&\times\left(1-\theta_{13}P_3(t)\right)\big(m_1\left(P_3(t)-c_3(t)\right)+\lambda_3 e^{rt}\big) \\
&-e^{-rt}q_{12}^2\left(1-f_{22}(t)-f_{23}(t)\right)\left(1-\theta_{22}P_2(t)\right)\times \\
&\big(m_2\left(P_2(t)-c_2(t)+r_{23}\left(P_3(t)-c_3(t)-D_{23}(t)\right)\right)+\lambda_4 e^{rt}\big) \\
&-e^{-rt}b_{23}^2 r_{23}(t)\left(1-f_{22}(t)-f_{23}(t)\right)\left(1-\theta_{23}P_3(t)\right)\big(m_2\left(P_3(t)-c_3(t)\right)+\lambda_5 e^{rt}\big) \\
&-e^{-rt}b_{23}^3 r_{23}(t)\left(1-f_{33}(t)\right)\left(1-\theta_{33}P_3(t)\right)\big(m_3\left(P_3(t)-c_3(t)\right)+\lambda_6 e^{rt}\big) \\
&-e^{-rt}m_1\big(\gamma_{23}+\beta_{13}^2 r_{13}(t)+\beta_{23}^2 r_{23}(t)\big)\left(1-r_{23}(t)\right)\phi_{23}D_{23}(t)\left(P_3(t)-c_3(t)-D_{23}(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_3 \;=\;& e^{-rt}\left(p_{11}+q_{11}f_{11}(t)\right)\left(1-\theta_{11}P_1(t)\right)\times \\
&\big(m_1\left(P_1(t)-c_1(t)+r_{12}\left(P_2(t)-c_2(t)-D_{12}(t)\right)+r_{13}\left(P_3(t)-c_3(t)-D_{13}(t)\right)\right)+\lambda_1 e^{rt}\big) \\
&+e^{-rt}\big(p_{12}+q_{12}^1 f_{12}(t)+q_{22}^1 f_{22}(t)+b_{12}^1 r_{12}(t)\, f_{11}(t)\big)\left(1-\theta_{12}P_2(t)\right)\times \\
&\big(m_1\left(P_2(t)-c_2(t)+r_{23}\left(P_3(t)-c_3(t)-D_{13}(t)\right)\right)+\lambda_2 e^{rt}\big) \\
&-e^{-rt}\big(q_{13}^1\left(1-f_{11}(t)-f_{12}(t)-f_{13}(t)\right)- \\
&\big(p_{13}+q_{13}^1 f_{13}(t)+q_{23}^1 f_{23}(t)+q_{33}^1 f_{33}(t)+b_{13}^1 r_{13}(t)\, f_{11}(t)+b_{23}^1 r_{23}(t)\left(f_{12}(t)+f_{22}(t)\right)\big)\big) \\
&\times\left(1-\theta_{13}P_3(t)\right)\big(m_1\left(P_3(t)-c_3(t)\right)+\lambda_3 e^{rt}\big) \\
&-e^{-rt}q_{13}^2\left(1-f_{22}(t)-f_{23}(t)\right)\left(1-\theta_{23}P_3(t)\right)\big(m_2\left(P_3(t)-c_3(t)\right)+\lambda_5 e^{rt}\big) \\
&-e^{-rt}q_{13}^3\left(1-f_{33}(t)\right)\left(1-\theta_{33}P_3(t)\right)\big(m_3\left(P_3(t)-c_3(t)\right)+\lambda_6 e^{rt}\big)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_4 \;=\;& -e^{-rt}q_{22}^{1}\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)-f_{13}\left(t\right)\right)\left(1-\theta_{12}P_2\left(t\right)\right)\times \\
& \left(m_1\left(P_2\left(t\right)-c_2\left(t\right)+r_{23}\left(P_3\left(t\right)-c_3\left(t\right)-D_{23}\left(t\right)\right)\right)+\lambda_2 e^{rt}\right) \\
& -e^{-rt}b_{23}^{1}r_{23}\left(t\right)\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)-f_{13}\left(t\right)\right)\left(1-\theta_{13}P_3\left(t\right)\right)\left(m_1\left(P_3\left(t\right)-c_3\left(t\right)\right)+\lambda_3 e^{rt}\right) \\
& -e^{-rt}\left(q_{22}^{2}\left(1-f_{22}\left(t\right)-f_{23}\left(t\right)\right)-\left(p_{22}+q_{12}^{2}f_{12}\left(t\right)+q_{22}^{2}f_{22}\left(t\right)+b_{12}^{2}r_{12}\left(t\right)f_{11}\left(t\right)\right)\right)\times \\
& \left(1-\theta_{22}P_2\left(t\right)\right)\left(m_2\left(P_2\left(t\right)-c_2\left(t\right)+r_{23}\left(P_3\left(t\right)-c_3\left(t\right)-D_{23}\left(t\right)\right)\right)+\lambda_4 e^{rt}\right) \\
& -e^{-rt}\left(b_{23}^{2}r_{23}\left(t\right)\left(1-f_{22}\left(t\right)-f_{23}\left(t\right)\right)-\right. \\
& \left.\left(p_{23}+q_{13}^{2}f_{13}\left(t\right)+q_{23}^{2}f_{23}\left(t\right)+q_{33}^{2}f_{33}\left(t\right)+b_{13}^{2}r_{13}\left(t\right)f_{11}\left(t\right)+b_{23}r_{23}\left(t\right)\left(f_{12}\left(t\right)+f_{22}\left(t\right)\right)\right)\right) \\
& \times\left(1-\theta_{23}P_3\left(t\right)\right)\left(m_2\left(P_3\left(t\right)-c_3\left(t\right)\right)+\lambda_5 e^{rt}\right) \\
& -e^{-rt}b_{23}^{3}r_{23}\left(t\right)\left(1-f_{33}\left(t\right)\right)\left(1-\theta_{33}P_3\left(t\right)\right)\left(m_3\left(P_3\left(t\right)-c_3\left(t\right)\right)+\lambda_6 e^{rt}\right) \\
& -e^{-rt}m_2\left(\gamma_{23}+\beta_{13}^{2}r_{13}\left(t\right)+\beta_{23}^{2}r_{23}\left(t\right)\right)\left(1-r_{23}\left(t\right)\right)\phi_{23}D_{23}\left(t\right)\left(P_3\left(t\right)-c_3\left(t\right)-D_{23}\left(t\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_5 \;=\;& -e^{-rt}q_{23}^{1}\left(1-f_{11}\left(t\right)-f_{12}\left(t\right)-f_{13}\left(t\right)\right)\left(1-\theta_{13}P_3\left(t\right)\right)\left(m_1\left(P_3\left(t\right)-c_3\left(t\right)\right)+\lambda_3 e^{rt}\right) \\
& +e^{-rt}\left(p_{22}+q_{12}^{2}f_{12}\left(t\right)+q_{22}^{2}f_{22}\left(t\right)+b_{12}^{2}r_{12}\left(t\right)f_{11}\left(t\right)\right)\left(1-\theta_{22}P_2\left(t\right)\right)\times \\
& \left(m_2\left(P_2\left(t\right)-c_2\left(t\right)+r_{23}\left(P_3\left(t\right)-c_3\left(t\right)-D_{23}\left(t\right)\right)\right)+\lambda_4 e^{rt}\right) \\
& -e^{-rt}\left(q_{23}^{2}\left(1-f_{22}\left(t\right)-f_{23}\left(t\right)\right)-\right. \\
& \left.\left(p_{23}+q_{13}^{2}f_{13}\left(t\right)+q_{23}^{2}f_{23}\left(t\right)+q_{33}^{2}f_{33}\left(t\right)+b_{13}^{2}r_{13}\left(t\right)f_{11}\left(t\right)+b_{23}r_{23}\left(t\right)\left(f_{12}\left(t\right)+f_{22}\left(t\right)\right)\right)\right) \\
& \times\left(1-\theta_{23}P_3\left(t\right)\right)\left(m_2\left(P_3\left(t\right)-c_3\left(t\right)\right)+\lambda_5 e^{rt}\right) \\
& -e^{-rt}q_{23}^{3}\left(1-f_{33}\left(t\right)\right)\left(1-\theta_{33}P_3\left(t\right)\right)\left(m_3\left(P_3\left(t\right)-c_3\left(t\right)\right)+\lambda_6 e^{rt}\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_6 &= -e^{-rt} q_{33}^1 \left(1 - f_{11}(t) - f_{12}(t) - f_{13}(t)\right) \left(1 - \theta_{13} P_3(t)\right) \left(m_1 \left(P_3(t) - c_3(t)\right) + \lambda_3 e^{rt}\right) \\
&\quad - e^{-rt} q_{33}^2 \left(1 - f_{22}(t) - f_{23}(t)\right) \left(1 - \theta_{23} P_3(t)\right) \left(m_2 \left(P_3(t) - c_3(t)\right) + \lambda_5 e^{rt}\right) \\
&\quad - e^{-rt} \Big(q_{33}^3 \left(1 - f_{33}(t)\right) - \\
&\quad \Big(p_{33} + q_{13}^3 f_{13}(t) + q_{23}^3 f_{23}(t) + q_{33}^3 f_{33}(t) + b_{13}^3 r_{13}(t) f_{11}(t) + b_{23}^3 r_{23}(t) \left(f_{12}(t) + f_{22}(t)\right)\Big)\Big) \\
&\quad \times \left(1 - \theta_{33} P_3(t)\right) \left(m_3 \left(P_3(t) - c_3(t)\right) + \lambda_6 e^{rt}\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mu}_1 &= -e^{-rt} \Big(\beta_{12}^1 \left(1 - r_{12}(t) - r_{13}(t)\right) - \left(\gamma_{12} + \beta_{12}^1 r_{12}(t)\right)\Big) \phi_{12} D_{12}(t) \times \\
&\quad \left(f_{11} m_1 \left(P_2(t) - c_2(t) - D_{12}(t)\right) + \mu_1 e^{rt}\right) \\
&\quad + e^{-rt} \left(\gamma_{13} + \beta_{13}^1 r_{13}(t) + \beta_{23}^1 r_{23}(t)\right) \phi_{13} D_{13}(t) \left(f_{11} m_1 \left(P_3(t) - c_3(t) - D_{13}(t)\right) + \mu_2 e^{rt}\right) \\
&\quad - e^{-rt} \left(p_{11} + q_{11} f_{11}(t)\right) \left(1 - f_{11}(t) - f_{12}(t) - f_{13}(t)\right) \left(1 - \theta_{11} P_1(t)\right) \times \\
&\quad m_1 \left(P_2(t) - c_2(t) - D_{12}(t)\right) \\
&\quad - e^{-rt} b_{12}^1 f_{11}(t) \left(1 - f_{11}(t) - f_{12}(t) - f_{13}(t)\right) \left(1 - \theta_{12} P_2(t)\right) \times \\
&\quad \left(m_1 \left(P_2(t) - c_2(t) + r_{23} \left(P_3(t) - c_3(t) - D_{23}(t)\right)\right) + \lambda_2 e^{rt}\right) \\
&\quad - e^{-rt} b_{12}^2 f_{11}(t) \left(1 - f_{22}(t) - f_{23}(t)\right) \left(1 - \theta_{22} P_2(t)\right) \times \\
&\quad \left(m_2 \left(P_2(t) - c_2(t) + r_{23} \left(P_3(t) - c_3(t) - D_{23}(t)\right)\right) + \lambda_4 e^{rt}\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mu}_2 \;=\; & e^{-rt}\left(\gamma_{12}+\beta_{12}^1 r_{12}(t)\right)\phi_{12}D_{12}(t)\left(f_{11}m_1\left(P_2(t)-c_2(t)-D_{12}(t)\right)+\mu_1 e^{rt}\right) \\
& -e^{-rt}\left(\beta_{13}^1\left(1-r_{12}(t)-r_{13}(t)\right)-\left(\gamma_{13}+\beta_{13}^1 r_{13}(t)+\beta_{23}^1 r_{23}(t)\right)\right)\phi_{13}D_{13}(t)\times \\
& \left(f_{11}m_1\left(P_3(t)-c_3(t)-D_{13}(t)\right)+\mu_2 e^{rt}\right) \\
& -e^{-rt}\beta_{13}^2\left(1-r_{23}(t)\right)\phi_{23}D_{23}(t)\left(\left(f_{12}m_1+f_{22}m_2\right)\left(P_3(t)-c_3(t)-D_{23}(t)\right)+\mu_3 e^{rt}\right) \\
& -e^{-rt}\left(p_{11}+q_{11}f_{11}(t)\right)\left(1-f_{11}(t)-f_{12}(t)-f_{13}(t)\right)\left(1-\theta_{11}P_1(t)\right)\times \\
& m_1\left(P_3(t)-c_3(t)-D_{13}(t)\right)- \\
& e^{-rt}b_{13}^1 f_{11}(t)\left(1-f_{11}(t)-f_{12}(t)-f_{13}(t)\right)\left(1-\theta_{13}P_3(t)\right)\left(m_1\left(P_3(t)-c_3(t)\right)+\lambda_3 e^{rt}\right) \\
& -e^{-rt}b_{13}^2 f_{11}(t)\left(1-f_{22}(t)-f_{23}(t)\right)\left(1-\theta_{23}P_3(t)\right)\left(m_2\left(P_3(t)-c_3(t)\right)+\lambda_5 e^{rt}\right) \\
& -e^{-rt}b_{13}^3 f_{11}(t)\left(1-f_{33}(t)\right)\left(1-\theta_{33}P_3(t)\right)\left(m_3\left(P_3(t)-c_3(t)\right)+\lambda_6 e^{rt}\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mu}_3 \;=\; & -e^{-rt}\beta_{23}^1\left(1-r_{12}(t)-r_{13}(t)\right)\phi_{13}D_{13}(t)\left(f_{11}m_1\left(P_3(t)-c_3(t)-D_{13}(t)\right)+\mu_2 e^{rt}\right) \\
& -e^{-rt}\left(\beta_{23}^2\left(1-r_{23}(t)\right)-\left(\gamma_{23}+\beta_{13}^2 r_{13}(t)+\beta_{23}^2 r_{23}(t)\right)\right)\phi_{23}D_{23}(t)\times \\
& \left(\left(f_{12}m_1+f_{22}m_2\right)\left(P_3(t)-c_3(t)-D_{23}(t)\right)+\mu_3 e^{rt}\right) \\
& -e^{-rt}\left(\left(p_{12}+q_{12}^1 f_{12}(t)+q_{22}^1 f_{22}(t)+b_{12}^1 r_{12}(t)f_{11}(t)\right)\left(1-f_{11}(t)-f_{12}(t)-f_{13}(t)\right)\times\right. \\
& \left(1-\theta_{12}P_2(t)\right)m_1+\left(1-\theta_{22}P_2(t)\right)m_2\times \\
& \left.\left(p_{22}+q_{12}^2 f_{12}(t)+q_{22}^2 f_{22}(t)+b_{12}^2 r_{12}(t)f_{11}(t)\right)\left(1-f_{22}(t)-f_{23}(t)\right)\right)\times \\
& \left(P_3(t)-c_3(t)-D_{23}(t)\right) \\
& -e^{-rt}b_{23}^1\left(f_{12}m_1+f_{22}m_2\right)\left(1-f_{11}(t)-f_{12}(t)-f_{13}(t)\right)\left(1-\theta_{13}P_3(t)\right)\times \\
& \left(m_1\left(P_3(t)-c_3(t)\right)+\lambda_3 e^{rt}\right) \\
& -e^{-rt}b_{23}^1\left(f_{12}m_1+f_{22}m_2\right)\left(1-f_{22}(t)-f_{23}(t)\right)\left(1-\theta_{23}P_3(t)\right)\times \\
& \left(m_2\left(P_3(t)-c_3(t)\right)+\lambda_5 e^{rt}\right) \\
& -e^{-rt}b_{23}^2\left(f_{12}m_1+f_{22}m_2\right)\left(1-f_{33}(t)\right)\left(1-\theta_{33}P_3(t)\right)\left(m_3\left(P_3(t)-c_3(t)\right)+\lambda_6 e^{rt}\right)
\end{aligned}
$$

For the scenario when the upgrade-rebate program is not implemented, the optimal control problem is a special case of the previous example where

$$
\begin{aligned}
\dot{r}_{12} &= 0, \dot{r}_{13} = 0, \dot{r}_{23} = 0 \\
D_{12} &= 0, D_{13} = 0, D_{23} = 0 \\
\dot{\mu}_1 &= 0, \dot{\mu}_2 = 0, \dot{\mu}_3 = 0
\end{aligned}
$$

Chapter 4

Dynamic Crossed Effects between Affiliate Marketing, Direct Access and Organic Search Traffic, Sales and Revenues

4.1 Introduction

Online promotion has been growing steadily during the past decade. Internet advertising revenues in the USA alone more than doubled from $12 billion total in 2005 to $26 billion in 2010, reaching a 15% growth in the last year (Interactive Advertising Bureau, 2011). Affiliate marketing, a powerful online advertising tool exploiting Word-of-Mouth (WOM), has become an important source of customer acquisition for many online retailers (Hoffman and Novak, 2000; Libai et al., 2003). Affiliate marketing programs consist of an online retailer placing a link on a third party (i.e. affiliates') website. A network of affiliate organizations can direct potential customers into a merchant's web page receiving referral fees in return. There are two types of pricing models. A commonly used mechanism is called "pay-per-lead", when the advertiser only pays for leads delivered that does not imply a purchase of the product the payment is done regardless whether the referrals are converted into buyers or not. Another type of pricing strategy is when the advertiser pays the affiliates a referral fee for every referral that has been converted into a buyer, and is called “pay-per-conversion”. When a user clicks on the link on the affiliate’s website and is directed to the advertiser’s web page, then the resulting purchase is attributed to the affiliate company, who receives some commission. Only these direct visits that generated revenues are attributed to affiliate marketing when evaluating affiliates’ effectiveness. Referral reward programs gain more and more importance as a means for customer acquisition. From a historic perspective, affiliate marketing was pioneered by William J. Tobin, the founder of PC Flowers & Gifts, in 1989. In 1994, CDnow launched its BuyWeb program. Shortly after, in 1996, Amazon.com launched its Associates program. Nowadays, many websites draw profits from affiliate programs. In fact, Forrester Research estimates that affiliate marketing spending in the United States will reach $4 billion by 2014, with annual growth rate of 16%.

Online retailers often wonder if referrals from affiliate marketing cannibalize sales from different online traffic sources, such as direct sales and converted organic search. Reciprocally, do direct and organic online clicks and sales affect the performance of affiliate marketing webs? The WOM spread by referral sites may strengthen consumers awareness, stimulating direct access and organic searches. Quantifying these effects is crucial: if affiliate marketing falls, which is the impact on direct and organic-search sales? And reciprocally, do affiliate websites draw clicks/sales from non-converted previous organic searches? These relationships must be studied from a dynamic perspective, as consumers can access through any of these alternative online sources, decide to postpone their purchases, and return later for subsequent purchases through other web traffic sources. Besides, the indirect effects between clicks, sales and revenues coming from each source can be different, implying that we must study all of these variables to have the full picture.

The study of these dynamic relationships must include all the affiliate sources. Affiliate companies vary in the volume of their operations, contribution to the advertiser's online sales (Ray, 2001), the degree of commonality of own and advertiser's products (Papatala and Bhatnagar, 2002). Because affiliate webs are so heterogenous, the effect on advertiser's sales and revenues is likely to be quite heterogeneous across affiliates. This requires to estimate a disaggregated large dimensional vector time series model. Given the large number of affiliates, the estimation a standard vector time series model, such as Vector AutoRegression (VAR) or a Vector Error Correction (VEC) models, is challenging. Instead, we employ a dynamic factor model suitable for large dimensions. In particular we follow the Global Vector AutoRegressive (GVAR) analysis considered by Pesaran et al. (2004). While common factors provide guidance on overall trends, results for individual affiliates may vary. Based on the dynamic factor model, we build a global VAR model for a better understanding of the crossed dynamic effects between each variable, computing the impulse response functions (IRFs) from this model.

We present an empirical application, based on data from an online retailer of jewelry. Jewelry is durable product, and our analysis focuses on long run effects. We therefore model cumulative sales, clicks and revenues. Our findings reflect in detail the dynamic forces shaping the affiliate marketing industry. The results of Impulse Response Function analysis show significant long term effect of affiliate marketing that clearly differs across affiliates. It is also important to note that the effects may be different for clicks, conversions, and revenues.

The rest of the paper is structured as follows. First, we provide a brief literature review. The third section describes the research framework and modeling approach. We then present our data set and empirical findings, and discuss the implications of our results for affiliate marketing. We finish with conclusion, limitations and future research steps.

4.2 Literature review

The marketing literature has studied how the consumer response to advertising can be magnified through WOM (Arndt, 1967; Hogan, Lemon and Libai, 2004). In the context of customer referrals, the existing literature also provides guidance to when customer referrals should be used (e.g. Biyalogorsky et al, 2001), analyzing the monetary value of a referral (Kumar, Petersen, and Leone, 2010) and the differential value between referred and non-referred customers (Schmitt et al., 2011). Marketing researchers have also started to study online business-to-business referrals. Libai et al. (2003) analyze why there exist both payment methods (pay-per-conversion and pay-per-lead) in affiliation marketing. Cai and Chen (2011) identify at least two forms of in-store referrals: a retailer may display the links to the competing retailers directly (direct referral), or display the referral link provided by a third-party advertising agency (third-party referral), both of which can be either one-way or two-way. Cai and Chen (2011) analytically study when the retailer should adopt referrals and compare the different forms of in-store referrals. Using survey data, Papatala and Bhatnagar (2002) study the choice of the right mix of affiliate companies. They find that the decision is going to depend on the relationship of the products that offer the affiliates and the advertiser: strict substitutes, strict complements, episodic substitutes, and episodic complements. Taking the affiliate's perspective, Akcura (2010) studies analytically the incentives of firms to join an affiliate marketing program diverting their customers to other websites. Still more research is needed to understand the performance of this type of programs.

Given the dynamic nature of WOM, the study of dynamic and long-term effects of affiliate marketing on performance of the effectiveness advertising rests on as a fundamental understanding of the question. First, the customers acquired by affiliates may later return to the online retailer's website through other channels of traffic for a subsequent purchase. Second, these customers might refer other potential buyers to the website through WOM. Affiliate networks might also improve the rankings of advertiser's web page in search engine results (Janssen and van Heck, 2007). Thus, some affiliates not only refer visitors and buyers to the retailer's website, but might also have long-term effects on the retailer's traffic, sales and revenues. On the other hand, however, affiliates may cannibalize the merchant's other marketing effort (i.e. they refer those customers who would visit the retailer's website in any case). Indirect effect have been found relevant in other contexts, such as paid search advertising. Using aggregate data of an e-commerce automotive company, Rutz et al. (2011) show that neglecting indirect effects of prior visits based on keyword searches can negatively bias paid search advertising conversion rates and upwardly bias sales revenues attributed to other traffic sources such as direct type-in and bookmark visitors.

This paper considers the dynamic effect of affiliate marketing on the advertiser's traffic, sales and revenues. We restrict our study to "pay-per-conversion" pricing strategy, although the analysis is valid under both payment arrangements. We investigate the effect of prior visits attributed to each affiliate company on the number of visits from direct access and bookmark, organic search, and other sources of traffic. In contrast to Rutz et al. (2011), we analyze the dynamic effect of affiliate marketing on not only the number of visits, but also on associated sales quantities and revenues to each channel.

4.3 Empirical Setting and Preliminary Analysis

Our empirical application is based on a weekly data set from an online retailer of jewelry. The data span 93 weeks, from January 2010 to October 2011. For this period of time we track the number of visits (clicks) to the retailer's website, the number of quantity purchased of the retailer's products and the associated revenues from each affiliate. We also observe the number of visits, purchased quantity and revenues disaggregated by the retailer's traffic sources: organic search, direct access and bookmark, and other.

In our sample period there are almost 800 affiliate companies associated through a large online advertising company, which have positive click-throughs or impressions, however only 181 of them bring at least one final buyer. We note that in our dataset a large part of revenues is generated by small group of affiliates, which is a common pattern in affiliate marketing (see, e.g., Ray, 2001). We let *AffiliateA* denote the aggregation of all small affiliates that individually bring less than 0.5% of purchased quantity during the sample period together accounting for more than 10% share of affiliate sales. Then, the remaining thirteen companies (denoted by *AffiliateB,...,AffiliateN*) provide nearly 90% of affiliate sales. Table 1 reports the total number of clicks, sales quantity and revenues coming from each affiliate during the sample period, with their corresponding shares.

Table 1. Affiliates' clicks, sales, revenues and their shares for the sample period

	Clicks		**Sales**		**Revenues(thousand$)**	
Variable	**Total**	**Share**	**Total**	**Share**	**Total**	**Share**
AffiliateA	188364	36.395	1359	10.56	211586.5	25.22
AffiliateB	22024	4.255	837	6.504	25102.1	2.99
AffiliateC	952	0.184	213	1.655	17513.5	2.09
AffiliateD	54939	10.615	1373	10.669	79134.7	9.43
AffiliateE	186240	35.984	7075	54.977	265749.2	31.68
AffiliateF	4068	0.786	126	0.979	13572.1	1.62
AffliateG	29937	5.784	445	3.457	50863.1	6.06
AffiliateH	6824	1.318	188	1.461	21937.5	2.61
AffiliateI	1486	0.287	203	1.577	33777.2	4.03
AffiliateJ	9603	1.855	129	1.002	15712.6	1.87
AffiliateK	8993	1.737	412	3.201	27092.9	3.23
AffiliateL	1406	0.271	237	1.842	26009.4	3.10
AffiliateM	1371	0.265	188	1.461	36898.0	4.39
AffiliateN	1347	0.260	84	0.652	13798.7	1.65
Total	517554	100	12869	100	838747.4	100

The weekly mean of the revenues brought up by each affiliate is higher than \$140. For the whole period, *Affiliate*A, which is aggregation of several small affiliates, has a high share of clicks but a low sales conversion rate. *Affiliate*E is the most profitable affiliate among all, accounting for \$21000 revenue for at least one week during the observation period. *AffiliateM* has a small clicks share of $0,26\%$, the conversion rate is so good that renders 1.46% of sales, and drives a $4,39\%$ of affiliates-driven revenues.

We also considered alternative traffic to the retailer's site: (1) Organic, coming from search engines (including Google, Yahoo, Bing, Nextag, Ask Jeeves, Aol search, msn search, and several other minor search engines), (2) direct type-in and bookmarked access, and (3) "other" sources (including traffic from special webpages such as happyannyversary.com, dealnews.com, social networks such as facebook.com, etc.). Regarding the data for each traffic source, the number of site visits by traffic sources is fairly equal for the sample period with direct type-in visits accounting for slightly less portion (31%) in the total number of visits. However, the pattern is different for sales quantity and revenues, "other" source accounting for the larger portion of sales and associated revenues than "organic" traffic from search engines (see Table 2).

Table 2. Traffic source clicks, sales, revenues and their shares for the sample period

	Clicks		**Sales**		**Revenues (thousand \$)**	
	Total	**Share**	**Total**	**Share**	**Total**	**Share**
Organic	395190	33.65	5196	11	924021	20
Direct	372886	31.75	14716	31	1617468	33
Other	406319	34.59	27515	58	2255320	47
Total	1174395	100	47427	100	4796809	100

As exogenous global variables for all the traffic sources we consider three variables: weekly average prices of gold and silver per troy ounce in US dollars, and a weekly measure of jewelry search intensity in Google. The latter measure shows how many searches have been done in Google where the term "jewelry" is present. This variable shows how many searches have been done for a period of time relative to total number of Google searches. These data are scaled to $0-100$ basis, i.e. the number of searches for each week is divided on the biggest number of searches during the year and is multiplied by 100. In fact, the search intensity variable achieves its the maximum value of 100 during Christmas 2010.

The overall numbers of visits (clicks) to the retailer's site coming from organic search, direct access or bookmark, and other sources show a seasonal increment starting two weeks prior to Christmas 2010 and a reduction of activity during February. A similar pattern is present also for the sales quantity and revenues from the retailer's each traffic source, as well as for the affiliate data series. Instead of applying intervention analysis in the time series model (with deterministic dummies associated to special weeks), we use a more related exogenous series. In particular we use a standardized variable of weekly search activities of term "jewelry" from Google Insights. This series is associated with the seasonal peaks, and therefore provides a good basis to forecast any sudden increment or drop due to calendar effects.

Let us stack all the considered endogenous variables in a time series vector $\{X_t\}$. Before considering a multivariate time series model, we examine the stationarity of the weekly series. Then, we say that a time series $\{X_t\}$ is integrated of order $d \in \{0, 1, 2, ..\}$, also denoted as $I(d)$, if *each coordinate* in $\Delta^d X_t$ follows an invertible stationary linear model, where $\Delta^d = (1 - L)^d$ and L is the lag operator ($L^j X_t = X_{t-j}$). Integrated series are said to have unit roots. In line with Deese et al. (2007), we perform Augmented Dickey-Fuller unit root test and weighted symmetric (WS) estimation of ADF type regressions proposed by Park and Fuller (1995). We use Schwartz Bayesian criterion (SBC) to select the lag length employed in unit root test. For each variable, we perform two unit root test regression: one including an intercept and a trend, and another including an intercept only. We also looked at the graphs and the autocorrelation functions. The results suggest that these series are stationary, particularly after removing the seasonal effects. However, the price of gold and silver are found to be I(1), meanwhile the standardized search intensity variable is stationary (see Table 3). Notice that I(0) and I(1) series cannot be related by a stable filter. Furthermore, the fact that the series are stationary, makes impossible to find permanent effects between these variables.

Table 3. Unit root test for common observed variables

	with trend		no trend		
Variables	**ADF**	**WS**	**ADF**	**WS**	**Conclusion**
Price of Gold	-3.34565	-3.11713	-0.53958	-0.36206	$I(1)$
Price of Silver	-2.52258	-2.40395	-1.18929	-1.18215	$I(1)$
Search Intensity	-3.99446	-4.18537	-3.84999	-4.01034	$I(0)$
Critical value	-3.45	-3.24	-2.89	-2.55	

Given the fact that jewelry is a durable product and our interest to explore persistent indirect effects, we have decided to model cumulative clicks, sales and revenues for affiliates and for the traffic sources, keeping the original gold and silver prices which are I(1) in levels. All the variables are now I(1). Given a vector $\{X_t\}$ of k time series, all of them $I(1)$, we say that the series are cointegrated if there are $r \leq k$ linear combinations defined by the $k \times r$ matrix β such that $\beta' X_t$ is jointly $I(0)$. Cointegrated series are typically modeled using error correction models, whereas non cointegrated series are modeled using a VAR for $\{\Delta X_t\}$. In the next section we build a factor model where the unobserved factors are allowed to be cointegrated. If there is cointegration, the specification process will lead us to a vector error correction model. If there is no cointegration, the specification process will direct us to stationary VAR model between the differentiated variables.

4.4 The Model

Our model will deal with N traffic sources. In particular we consider $N = 17$ given by the number of affiliates (in our case 14, after aggregating the small ones in *AffiliateA*), plus three additional key sources (organic search engines traffic, direct access and bookmark, and other sources). For each source $i = 1, ..., N$, at time t the retailer observes the cumulative number of clicks over time, the cumulative number of quantity purchased by customers attracted by each source and the associated cumulative revenues, which we stack in vector X_{it}. We denote by d_t the vector of observed global factors (in our case, price for gold, price of silver, and "jewelry" searches on internet). Notice that all these variables are $I(1)$. To overcome the dimensionality problem, we specify a dynamic factor model. In particular we build a GVAR model (Pesaran et al. 2004, Dees et al. 2007). Denote the unobserved factor by f_t, the model consider that for $i = 1, 2, \ldots, N$ and $t = 1, 2, \ldots, T$ it is satisfied that:

$$X_{it} = \delta_{i0} + \delta_{i1} t + \Gamma_{if}\ f_t + \Gamma_{id}\ d_t + \xi_{it}, \tag{4.1}$$

where $(\Gamma_{if}\ , \Gamma_{id})$ is the matrix of factor loadings, and ξ_{it} are traffic source specific random shocks. The intercept and trend coefficients δ_{i0}, δ_{i1} are associated to deterministic patterns. The model accommodates unit roots and cointegration relationships in X_{it} allowing the unobserved and observed global factors $h_t = (f_t', d_t')'$ to be $I(1)$ and to be cointegrated, considering two independent process:

$$\begin{aligned} \Delta h_t &= \sum_{l=0}^{\infty} \Lambda_l\ \eta_{t-l} = \Lambda(L)\ \eta_t, \\ \Delta \xi_{it} &= \sum_{l=0}^{\infty} \Psi_{il}\ \nu_{t-l} = \Psi_i(L)\ \nu_{it}, \end{aligned}$$

where $\Lambda(L), \Psi_i(L)$ are invertible polynomials in the lag operator L with absolutely summable coefficients, $\eta_t \sim IID(0, I)$ and $\nu_{it} \sim IID(0, I_i)$. Then $Var[\xi_{it}] = \sum_{l=0}^{\infty} \Psi_{il} \Psi_{il}'$ is bounded. To estimate this model, each traffic source is decoupled from the rest of the online sales sources. To that end we consider

$$\Phi_i(L) = (1 - L)\ \Psi_i(L)^{-1},$$

which can be approximated by a finite polynomial as the coefficients of $\Psi_i(L)^{-1}$ tend to zero at an exponential rate. In practice we consider p_i lags (in our data, the empirical analysis suggests $p_1 = 1$ for all traffic sources). Then we can rewrite the model as

$$\Phi_i(L)(X_{it} - \delta_{i0} - \delta_{i1}t - \Gamma_{if}\ f_t - \Gamma_{id}\ d_t) = \xi_{it}. \tag{4.2}$$

where d_t is weakly exogenous. This model can be estimated independently for each traffic source, computing a weighted average $X_t^* = \sum_{j=1}^{N} w_j X_{jt}$, where $w_j \geq 0$ add up to one, which is a process satisfying

$$X_t^* = \delta_0^* + \delta_1^* t + \Gamma_f^*\ f_t + \Gamma_d^*\ d_t + \xi_t^*,$$

where the parameters are weighted means of the original ones. Next, we introduce some limit regularity on the traffic sources. We assume that as N grows the parameter means are stable, and for moderate N do not depend on i. Under regularity conditions $\Delta\xi_t^* = \xi_t^* - \xi_{t-1}^* = \sum_{j=1}^{N} w_j \Psi_j(L)\ \nu_{jt}$ tends to zero fast in mean square sense when $N \to \infty$ suggesting that ξ_t^* approximates a time invariant random variable ξ^*. The procedure also allows a specific means for each traffic source, considering weights $\{w_{ij}\}_{j=1}^{N}$ and setting $w_{ii} = 0$ for a finite N. Using this idea and the assumption that $\overline{\Gamma}_{if}$ has full rank, the unobserved factor

$$f_t \to \left(\Gamma_f^{*\prime}\Gamma_f^*\right)^{-1}\Gamma_f^*\left(X_t^* - \delta_0^* - \delta_1^* t - \Gamma_d^*\ d_t - \xi^*\right)$$

in a mean squared sense, and substituting these approximations in the model we obtain

$$\Phi_i(L)\left(X_{it} - \widetilde{\delta}_{i0} - \delta_{i1}t - \widetilde{\Gamma}_{id}\ d_t - \widetilde{\Gamma}_{if}\ X_{it}^*\right) = u_{it}. \tag{4.3}$$

Without loss of generality, we can modify the lags in the observed exogenous factors d_t.

4.4.1 Cointegration analysis per traffic source

In the model we allow for cointegration in the original $I(1)$ series. Substituting $\Phi_i(L) = (1-L)\Psi_i(L)^{-1}$ in (4.3), and expanding the matrix lag polynomial $\Psi_i(L)$ in a Taylor expansion around $L=1$, we obtain

$$\begin{aligned}\left(X_{it} - \widetilde{\delta}_{i0} - \delta_{i1}t - \widetilde{\Gamma}_{id}\ d_t - \widetilde{\Gamma}_{if}\ X_{it}^*\right) &= (1-L)^{-1}\Psi_i(L)u_{it} \\ &= (1-L)^{-1}\left(\Psi_i(1) - \Psi_i^{1)}(L)(1-L)\right)u_{it},\end{aligned}$$

where the roots of $\left|\Psi_i^{1)}(L)\right|$ are outside the unit circle, and the term $(1-L)^{-1}\Psi_i(1)u_{it}$ represents an stochastic trend. Cointegration between these series means that $\beta'\left(X_{it} - \widetilde{\delta}_{i0} - \delta_{i1}t - \widetilde{\Gamma}_{id}\ d_t - \widetilde{\Gamma}_{if}\ X_{it}^*\right)$ is $I(0)$ for a matrix β with full rank, which happens if and only if the matrix $\Psi_i^{1)}(1)$ satisfy $\beta'\Psi_i^{1)}(1) = 0$. The linear combination must be interpreted as long-run equilibrium relationships between the original $I(1)$ series removed from the weakly exogenous and deterministic components. The rank of $\Psi_i^{1)}(1)$ determines the number of cointegration relationships in β.

Similar arguments can be used to derive the vector error correction representation. Since $\Phi_i(L)\Psi_i(L) = (1-L)I$, then $\Phi_i(1)\Psi_i^{1)}(1) = 0$ implying that the rows of $\Phi_i(1)$ belong to the linear space defined by $\beta'\Psi_i^{1)}(1) = 0$ and we can express $\Phi_i(1) = \alpha_i\beta_i'$. Taking a first order Taylor expansion of $\Phi_i(L)$, and also adding and substracting $\Phi_i(1)L$ we obtain that

$$\Phi_i(L) = \Phi_i(1) + \Phi_i^{1)}(L)(1-L) = \left(\alpha_i\beta_i'L + \left(\alpha_i\beta_i' + \Phi_i^{1)}(L)\right)(1-L)\right).$$

Substituting this expression in (4.3), renders the Granger's vector error correction representation.

We conduct the empirical analysis making use of the Matlab GVAR Toolbox 1.1. (Smith and Galesi, 2011). We first conduct a cointegration analysis to identify whether a long-run equilibrium exists among the series (i.e. the series are cointegrated). We first select the order of individual traffic source VARX models, using Schwartz-Bayesian information criterion. Afterwards we compute maximum eigenvalue and trace statistics, used to determine the cointegration space for individual models. This analysis suggests to use VECX$(1, 1)$ estimation with $r = 1$ (one cointegration relationship) for each individual model. The individual models are estimated following Johansen's method. The main assumption for estimating the individual traffic source models is that X_{it}^* is weakly exogenous with respect to the long run parameters of the conditional VEC model (for details see Smith and Galesi, 2011, pp. 92).

Before conducting the dynamic analysis, we further diagnose the main assumptions which underlie our GVAR modeling approach. We first test weak exogeneity of X_{it}^* with respect to long run parameters of VECX estimation. For all the individual models, the weak exogeneity assumption was only rejected for 7 out of 102 tests at the 5% significance level. Further, we check the descriptive statistics and Jarque-Bera normality test for the residuals obtained from VECX estimation. We also inspect the serial correlation of VECX residuals based on Lagrange Multiplier serial correlation test with F statistics. We observe that we fail to reject VECX residual serial correlation only for 7 cases out of 54 tests.

4.4.2 Global VAR Model

Dynamic models with unobserved factors are powerful tools for the analysis of dynamic relationships between a large number of variables. The dynamic factor model provides guidance on overall underlying trends, but given the heterogeneity of online traffic drawn from different sources it is convenient to build a global VAR model for an interpretation of co-movements between the different series. In particular, the estimated parameters of dynamic factor model can be used to build a global VAR model to compute the impulse response functions showing crossed dynamic effects between the considered variables. We stack all these variables in a column vector $X_t = (X'_{1t}, ..., X'_{Nt})'$. This is a large dimensional vector of $I(1)$ variables. Noticing that the weighted average can be expressed as $X^*_{it} = W_i X_t$, then all of the individual traffic source equations can be grouped to build a global structural VAR model,

$$G(L) X_t = \varphi_t + u_t,$$

where φ_t is a column vector which stacks the deterministic trend and the observable exogenous factors. This model is identified, since the weights W_i are known and the individual submodels are indentified. In particular for $p = 1$ we obtain a GVAR(1) model $G_0 X_t = \varphi_t + G_1 X_{t-1} + u_t$. Premultiplying both sides of the equation by G_0^{-1} and changing the notation, we obtain the GVAR(1) in the reduced form

$$X_t = G_0^{-1} (\varphi_t + G_1 X_{t-1} + u_t) = \phi_t + F_1 X_{t-1} + \upsilon_t. \tag{4.4}$$

The latter can be udated recursively starting from an initial value X_0, and used for impulse response and forecasting analysis, which we perform in the following section.

4.5 Main results

In this section we discuss the short and long term effects that we have found after estimating the model. First we estimate the parameters for each submodel, which are significant and we perform diagnosis tests over the main exogeneity and lags specification assumptions. Then the traffic source submodels are integrated in a global VAR. We do not include the estimators, given the large dimension of this table, and the discussion will focus on the interpretation of direct and indirect effects.

4.5.1 Contemporaneous effects

We first study the contemporaneous effects across the variables. Consider the model $X_t = \phi_t + F_1 X_{t-1} + \upsilon_t$. Note that the elements of the matrix $Var(\upsilon_t) = G_0^{-1} Var(u_t) G_0^{\prime -1}$ contain the contemporary effects across the variables in the global model. This matrix contains contemporary effects of cumulative clicks, sales and revenues both within each traffic source and across one traffic source of the other. By contemporary effects we mean association of responses in several variables within the same week.

We observe that all the elements of the matrix $Var(\upsilon_t)$ are positive. Notice that pre and postmultiplying $Var(\upsilon_t)$ by the inverse of the diagonal matrix $Diag\left\{\sqrt{Var(\upsilon_{it})}\right\}$ containing the standard deviations of the elements in υ_t we obtain the correlation matrix. The off-diagonal elements of the matrix show the correlation between the elements of the vector υ_t. Thus, the positive sign and size of the correlations suggest direct relationships among cumulative clicks, sales, and revenues across all the the traffic sources of the online retailer. In particular, in Table 4 we report the contemporary effects (correlations) between cumulative clicks, sales, and revenues from direct access and organic search with the variables from *AffiliateE* and *AffiliateJ*. We note that the positive contemporaneous cross relationships could be due to positive higher awareness of the retailer's website and positive contemporaneous WOM spread by the buyers.

Table 4. Contemporaneous cross effects

	AffiliateE			*AffiliateJ*		
	Clicks	**Sales**	**Revenues**	**Clicks**	**Sales**	**Revenues**
Direct Access						
Clicks	0.58	0.31	0.64	0.56	0.56	0.54
Sales	0.61	0.32	0.67	0.55	0.56	0.57
Revenues	0.57	0.28	0.63	0.52	0.55	0.56
Organic Search						
Clicks	0.45	0.22	0.49	0.51	0.50	0.47
Sales	0.51	0.27	0.63	0.52	0.51	0.51
Revenues	0.40	0.24	0.57	0.57	0.47	0.50

4.5.2 Dynamic effects

We then turn to study the dynamic relationships between cumulative clicks, sales, and revenues from the online retailer's different sources of traffic using (4.4). We first note that from the coefficient matrix in the GVAR(1)

$$X_t = \phi_t + F_1 X_{t-1} + \upsilon_t. \tag{4.5}$$

one can get direct effects of cumulative clicks, sales, and revenues at $(t-1)$ on the same variables at time t within and across each traffic source. The direct effects of different affiliate companies are heterogenous. However, the impact of an increment in any of these variables is transmitted indirectly to $X_{t+1}, X_{t+2}, ...$ In order to obtain a more complete understanding we need to study the indirect effects provided by the impulse response functions.

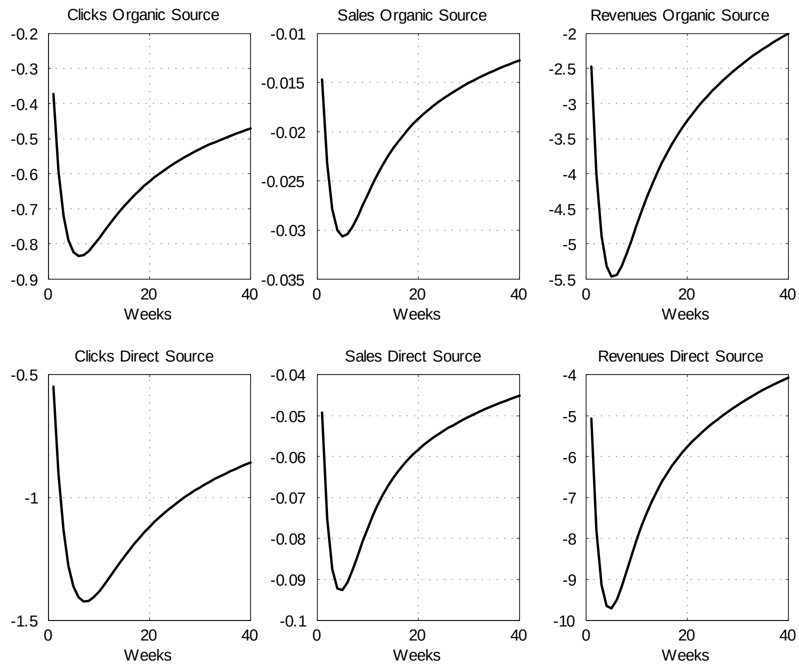
In order to present the dynamic relationships between the series, we conduct impulse response analysis. Note that the model (4.5) can be normalized premultiplying by $Var(\upsilon_t)^{1/2}$ to obtain orthogonal IRFs (as it is sometimes done in applied macroeconomics), or one can compute the classical IRFs without any normalization (it is the standard procedure in the time series literature) which has a more direct interpretation. In this paper we follow the second option, extracting the IRFs from the global VAR autoregressive polynomial $F(L) = (I - F_1 L)$. Notice that the inverse of $F(L)$ is not convergent, as we have common stochastic trends, but we can apply cumulatively $F(L)^{-1}$ to error impulse shocks, and show how the effects are transmitted. In particular for a VAR(1) the impulse responses after l lags are given by the elements of the power matrixes of $(F_1)^l$. Note that some effects will disappear, but others will be persistent or even grow in time (as X_t are $I(1)$ variables) so that the cumulative effect of an impulse diverges for these variables.

We mostly focus on the dynamic effects on cumulative revenues. We first investigate the implications of a shock in cumulative clicks from affiliates on future cumulative clicks, cumulative sales quantity and cumulative revenues drawn from organic search source and direct type-in and bookmarked visits. We also examine the reverse effect, i.e. the impact of a past shock to cumulative clicks from organic search and direct access on cumulative clicks, cumulative sales quantity and cumulative revenues drawn from affiliates. Because of space limitations we do not report IRFs related to all of the affiliates. As an example, we focus on the dynamic relationship of traffic sources such as organic search and direct access with *AffiliateE* and *AffiliateJ*. As can be observed from Table 1, *AffiliateE* is the one providing a larger portion of affiliate clicks, rendering a large proportion of sales and revenues, but the conversion rate is not particularly good. On the other hand, *AffiliateJ* brought relatively few final buyers during the sample period. Figures 1 and 2 show the IRF plots for *AffiliateE* and *AffiliateJ*, respectively.

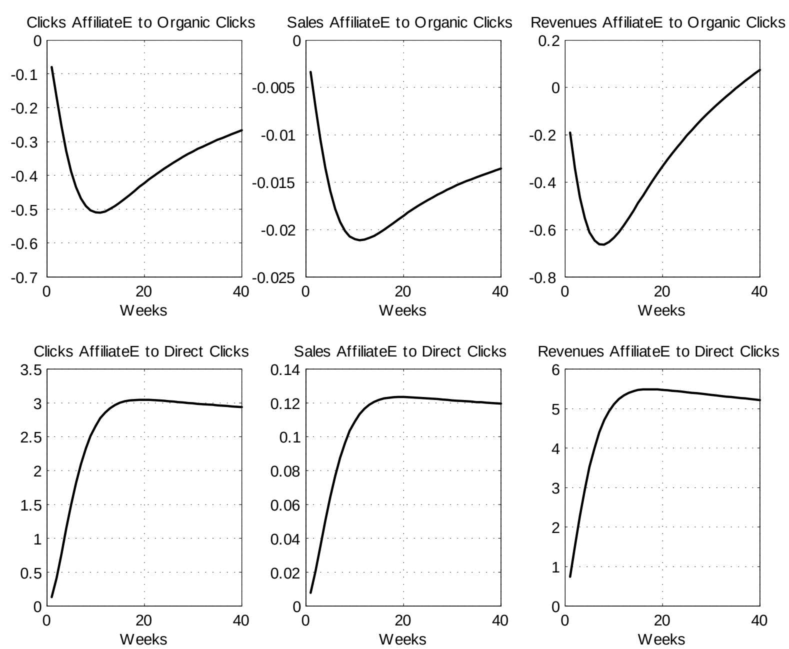
We observe that the impact of clicks from *AffiliateE* on clicks, sales and driven by organic search and direct access is negative, although the effects decay over time (Figure 1, a). The clear implication is that *AffiliateE* refers customers who would most likely visit the online retailer through organic search or direct type-in and bookmark access channels. However, the signs of the reverse effect are not always the same. In particular, the effect of clicks from organic search on clicks, sales and revenues from *AffiliateE* is negative, whereas in the case of direct type-in access is positive suggesting that some users of *AffiliateE* postpone their purchases typing in directly lager, and/or communicate the address through WOM (Figure 1, b).

Figure 1. Dynamic relationship between *AffiliateE* and organic and direct access sources

(a) Response to clicks from *AffiliateE*



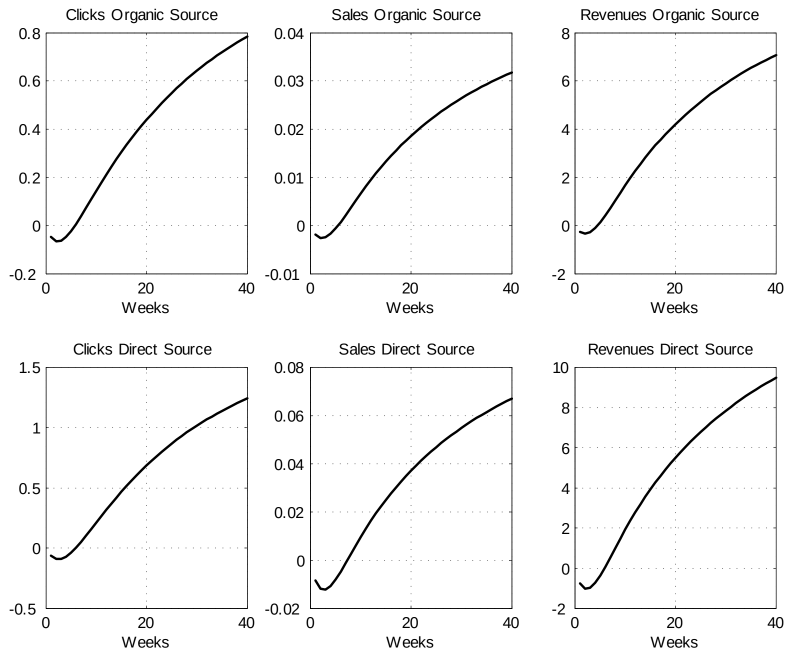
(b) Response of clicks, sales and revenues of *AffiliateE*



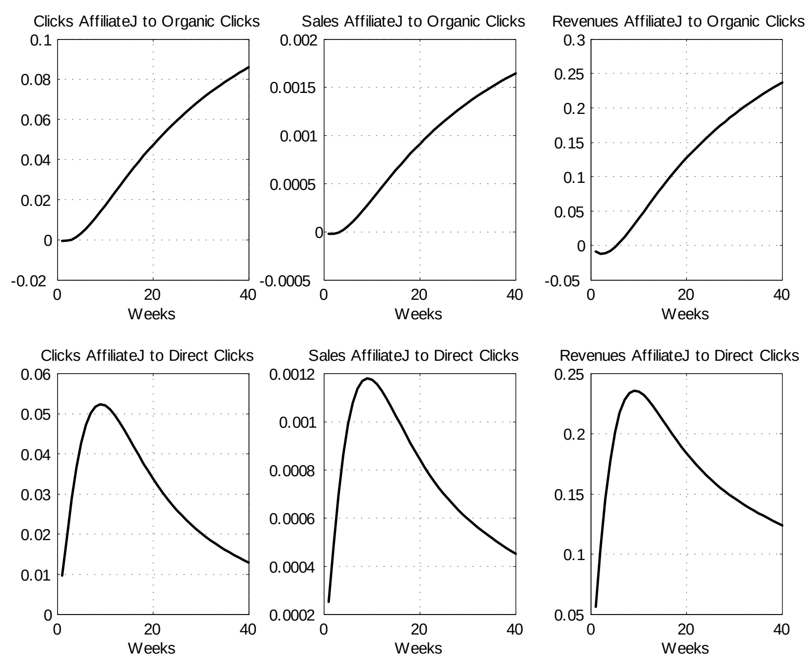
Turning to the dynamic relationship between *AffiliateJ* and the traffic sources of interest, we observe different patterns than in the case of *AffiliateE*. Specifically, the impact of clicks from *AffiliateJ* on clicks, sales and revenues from organic search and direct type-in access is only negative for an initial one-two weeks growing to positive in the long-run. Nevertheless, we note that this effect becomes positive earlier for the case of organic search channel than for direct type-in and bookmark access (Figure 2. a). In contrast with case of *AffiliateE*, the impact of clicks from organic search on clicks, sales and revenues of *AffiliateJ* is positive and growing over time. The impact of clicks from direct type-in access on the variables related to *AffiliateJ* is also positive, but decaying over time.

Figure 2. Dynamic relationship between *AffiliateJ* and organic and direct access sources

(a) Response to clicks from *AffiliateJ*



(b) Response of clicks, sales and revenues of *AffiliateJ*



Comparing Figures 1 and 2, we observe that the dynamic cross-relationships between clicks, sales and revenues between the different traffic sources can be very different. Studying the crossed long term effect of a past shock on clicks from each affiliate on the revenues from organic search and direct type-in access, we find that for 7 out of 14 affiliates the effect is negative and for the other 7 it is positive. Examining these relationships across all the affiliates, the general pattern found is that those clicks from affiliates which are more effective to conversion of clicks to sales and revenues have positive long term impact on sales and revenues from organic search and direct type-in channels.

We have examined the effect of past shocks on cumulative clicks from each affiliate on the revenues rendered by the same affiliate (own effects). Interestingly, for some affiliates the own effects are positive and for the others are negative. Overall, we observe four types of dynamic effects of clicks on revenues according to the own and cross effect between the different traffic sources: (i) **negative-positive**, if for an affiliate the long term effect of a shocks in the cumulative clicks on own-traffic cumulative revenues is negative, but positive on revenues from direct and organic search access, then the potential customers referred from this affiliate delay the decision returning later to the retailer either directly or through an online search engine, and/or referring to other potential buyers the webpage of the retailer (and not the affiliate); (2) **positive-positive**, positive-negative, in case both the cross and own long term effect on revenues are positive, we may consider that the visitors from the affiliate do not only refer to the other buyers the website of the online retailer, but also the website of the affiliate balancing the positive effect across these traffic sources; (3) **positive-negative**, when but the own effect on revenues is positive but the cross effect is negative, we postulate that the customers loyalty falls in the context of the affiliate, who in the future may refer the customer to other manufacturers in the future; and (4) **negative-negative**, in case both cross and own effects are negative, we postulate that this could be due to negative WOM spread by the visitors attracted by the affiliate, indicating that the affiliate perhaps transmitted too high expectations or a biased image. Sometimes the initial response falls in one of these types, but then it changes after a short period (as we observe in Figure 2.a).

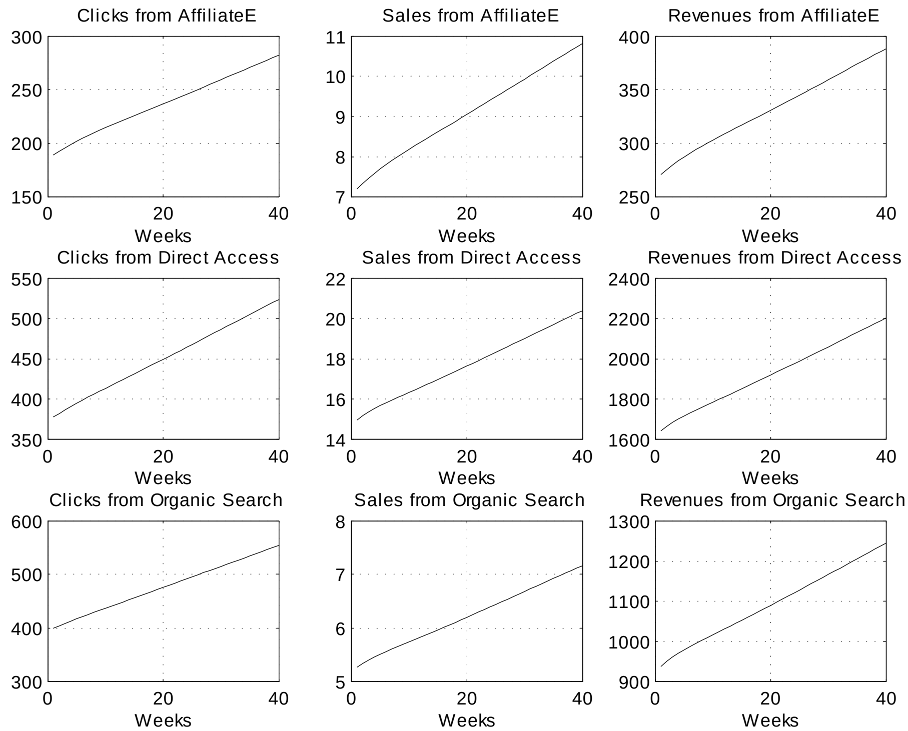
Why are these long term effects different across affiliates? One of the reasons could lie in the relationship between the products offered by the affiliates and the online retailer, brought up by Papatala and Bhatnagar (2002). Another reason could be the promotional tools employed by the affiliates. For instance, we find that *AffiliateM* has the highest long term impact on the sales and revenues from organic search and direct type-in access. From Table 3 it can be observed that *AffiliateM* is also the most effective affiliate in term sales and revenue conversion, with on average each seventh click being converters to a final purchase. We further verify with the managers of the online retailer that this affiliate is using *search engine marketing* as a main online marketing tool. Other reasons could be affiliate specific characteristics such as website design and structure, the number and heterogeneity of advertisers served by each affiliate, etc.

Unfortunately, there is not much research done towards explaining the reasons of different long terms effects among affiliates. Nevertheless, our findings reveal that affiliates are different not only in their conversion rates, but also in cross and own long term effects on other channels of traffic being negative for some cases. We suggest that the manager of the online retailer should be cautious especially of those affiliates who exhibit negative own and cross long term effects on revenues. The manager of the online retailer should carefully weight the positive contemporaneous benefits against the long term negative effects. Some affiliates can be prioritized or given specific incentives to generate traffic, or withdrawn from the portfolio depending on the long run revenues response.

4.5.3 Forecasting Analysis

We use the GVAR model solution for forecasting purposes. For an illustration, we plot forecasted number of cumulative clicks, cumulative sales quantity and revenues for duration of 40 weeks period from *AffiliateE*, direct type-in and organic search access.

Figure 3. Forecasted cumulative clicks, sales and revenues (in thousands)



It can be observed, that within the forecast period the cumulative number of clicks from *AffiliateE* increased by nearly 49%, cumulative sales quantity by 50%, meanwhile sales revenues by 43%. On the other hand, the cumulative number of clicks, sales, and revenues increased by around 39%, 36%, and 34%, respectively, both for organic search and direct type-in access. Similarly, the manager of the retailer could use the GVAR solution for forecasting cumulative clicks, sales and revenues from the other traffic sources.

The forecast analysis could be employed by the manager of the online retailer for several managerial insights about its affiliate marketing. It can be used for comparison of affiliate trends with each other. Forecast analysis can be also used to study the composition of total number of clicks, sales quantity and revenues from different traffic sources, as well as the forecasted contribution of each affiliate to the general clicks, sales quantity and corresponding revenues.

4.6 Conclusion

In this paper we attempted to look beyond the immediate effect of affiliates and examined affiliate marketing from dynamic point of view. The value of the visitors and final buyers attracted by the affiliates does not end with their initial visit and/or purchase. These online users may later return to the advertiser through other web channels. They could also refer other potential customers to the online retailer. Further, each affiliate may cannibalize the marketing effort of other affiliates and the retailer itself. Thus, the manager of an online retailer should aim to account for these dynamic forces shaping its affiliates network.

We aim to propose a modeling approach which enables a manager of an online retailer to study the long term impact of each of its affiliates on traffic, sales and revenues for each of its traffic sources. The modeling challenge is that typically the affiliate sites are heterogeneous and very large in number. We propose to solve this curse of dimensionality problem applying GVAR analysis developed by Pesaran et al (2004) and Dees et al (2007).

Our empirical analysis is based on data from an online retailer selling jewelry. We conduct a GVAR analysis and use the GVAR solutions for forecasting and impulse response analysis. We find that the effects of different shocks are heterogeneous across affiliates. We observe spill-over as well as cannibalization effects among affiliates. Overall, we think that GVAR analysis is an appropriate approach for a manager who seeks to study dynamic relationships among its traffic sources and affiliates. It is also a suitable tool to examine the long-term effectiveness of each affiliate website beyond the traditional affiliate performance measures (e.g. conversion rate).

Our study raises several interesting questions in affiliate marketing which could be addressed in future research. First, we do not explicitly model the structural changes associated with entry of a website to the affiliate network. In the spirit of Pesaran, Smith, and Smith (2007), future research could use the modeling framework to quantify the effects of "what if" scenarios when other company joins to the affiliate network of the retailer. Future research could also address dynamic relationship between affiliate marketing performance and other online advertising channels. Finally, additional research should analyze the consumer behavior after exposure to different types of affiliate webs, building a taxonomy of communication strategies and the link between the reactions and the long term performance effects quantified in this article.

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